Text Book for

INTERMEDIATE

First Year

Mathematics

Paper - IA

Algebra, Vector Algebra, Trigonometry

Telugu and Sanskrit Akademi

Andhra Pradesh

Intermediate
Second Year

Mathematics
Paper - IA
Text Book

Pages : xx + 456 + viii + iv

Reprint : 2023

Copies : 35000

Published, Printed & Distributed by
Telugu and Sanskrit Akademi, A.P.

Price: Rs. **236.00**

Laser Typeset by **Pavan Graphics, Hyderabad**

Published and Printed by
M/s GBR Offset Printers & Publishers
Surampalli, NTR Dist.

on behalf of **Telugu and Sanskrit Akademi**

Y.S. JAGAN MOHAN REDDY

CHIEF MINISTER
ANDHRA PRADESH

AMARAVATI

MESSAGE

I congratulate Akademi for starting its activities with printing of textbooks from the academic year 2021 – 22.

Education is a real asset which cannot be stolen by anyone and it is the foundation on which children build their future. As the world has become a global village, children will have to compete with the world as they grow up. For this there is every need for good books and good education.

Our government has brought in many changes in the education system and more are to come. The government has been taking care to provide education to the poor and needy through various measures, like developing infrastructure, upgrading the skills of teachers, providing incentives to the children and parents to pursue education. Nutritious mid-day meal and converting Anganwadis into pre-primary schools with English as medium of instruction are the steps taken to initiate children into education from a young age. Besides introducing CBSE syllabus and Telugu as a compulsory subject, the government has taken up numerous innovative programmes.

The revival of the Akademi also took place during the tenure of our government as it was neglected after the State was bifurcated. The Akademi, which was started on August 6, 1968 in the undivided state of Andhra Pradesh, was printing text books, works of popular writers and books for competitive exams and personality development.

Our government has decided to make available all kinds of books required for students and employees through Akademi, with headquarters at Tirupati.

I extend my best wishes to the Akademi and hope it will regain its past glory.

(Y.S. JAGAN MOHAN REDDY)

Dr. NANDAMURI LAKSHMIPARVATHi
M.A., M.Phil., Ph.D.
Chairperson, (Cabinet Minister Rank)
Telugu and Sanskrit Akademi, A.P.

Message of Chairperson, Telugu and Sanskrit Akademi, A.P.

In accordance with the syllabus developed by the Board of Intermediate, State Council for Higher Education, SCERT etc., we design high quality Text books by recruiting efficient Professors, department heads and faculty members from various Universities and Colleges as writers and editors. We are taking steps to print the required number of these books in a timely manner and distribute through the Akademi's Regional Centers present across the Andhra Pradesh.

In addition to text books, we strive to keep monographs, dictionaries, dialect texts, question banks, contact texts, popular texts, essays, linguistics texts, school level dictionaries, glossaries, etc., updated and printed and made available to students from time to time.

For competitive examinations conducted by the Andhra Pradesh Public Service Commission and for Entrance examinations conducted by various Universities, the contents of the Akademi publications are taken as standard. So, I want all the students and Employees to make use of Akademi books of high standards for their golden future.

Congratulations and best wishes to all of you.

N. Lakshmiparvathi

(NANDAMURI LAKSHMIPARVATHI)

J. SYAMALA RAO, I.A.S.,
Principal Secretary to Government

Higher Education Department
Government of Andhra Pradesh

MESSAGE

I Congratulate Telugu and Sanskrit Akademi for taking up the initiative of printing and distributing textbooks in both Telugu and English media within a short span of establishing Telugu and Sanskrit Akademi.

Number of students of Andhra Pradesh are competing of National Level for admissions into Medicine and Engineering courses. In order to help these students Telugu and Sanskrit Akademi consultation with NCERT redesigned their Textbooks to suit the requirement of National Level Examinations in a lucid language.

As the content in Telugu and Sanskrit Akademi books is highly informative and authentic, printed in multi-color on high quality paper and will be made available to the students in a time bound manner. I hope all the students in Andhra Pradesh will utilize the Akademi textbooks for better understanding of the subjects to compete of state and national levels.

(J. SYAMALA RAO)

THE CONSTITUTION OF INDIA

PREAMBLE

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a [SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC] and to secure to all its citizens:

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the [unity and integrity of the Nation];

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949 do HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.

Textbook Development Committee

Chief Coordinator

Prof. P.V. Arunachalam, Founder Vice-Chancellor, Dravidian University, Kuppam

Editors

Prof. K. Pattabhi Rama Sastry, Professor of Mathematics (Retd), Andhra University, Visakhapatnam

Prof. U.M. Swamy, Professor of Mathematics (Retd). Andhra University. Visakhapatnam

Prof. I. Ramabhadra Sarma, Professor of Mathematics (Retd), Acharya Nagarjuna University, Guntur

Prof. Y. Venkateswara Reddy, Professor of Mathematics(Retd.), Acharya Nagarjuna University, Guntur

Prof. D. Rama Murthy, Professor (Retd). Department of Mathematics. Osmania University. Hyderabad

Prof. K. Sambaiah, Chairman, BOS, Department of Mathematics, Kakatiya University. Warangal

Sri A. Padmanabham, Head of the Department of Mathematics (Retd.), Maharani College, Peddapuram

Sri Sonthi Krishna Sastry, Research Assistant (Retd), Mathematics, Telugu Akademi, Hyderabad

Authors

Sri K.V.S. Prasad, Lecturer in Mathematics, Govt. Degree College (Women), Srikakulam

Smt. S.V. Sailaja, Junior Lecturer in Mathematics, New Government Junior College, Kukatpally. Hyderabad

Sri V. Sreenivasulu, Lecturer in Mathematics, Govt. Degree College, Mahaboobabad, Warangal.

Sri G. Sethu Madhava Rao, Associate Professor of Mathematics, Kakatiya Govt. Degree College, Hanamkonda, Warangal.

Sri G. Suryanarayana Murthy, Reader in Mathematics (Retd). R.S.R.K Ranga Rao College, Bobbili

Prof. K. Rama Mohana Rao, Professor of Applied Mathematics (Retd). Andhra University, Visakhapatnam

Prof. G. Chakradhara Rao, Professor of Mathematics & Chairman, P.G. Board of Studies, Andhra University, Visakhapatnam

Sri K. Chandra Sekhara Rao, Junior Lecturer, Govt. Junior college , Uppugunduru. Prakasam (Dist)

Dr. D. Chitti Babu, Reader, Department of Mathematics, Govt. College (Autonomous), Rajahmundry

Sri C. Sadasiva Sastry, Lecturer in Mathematics (Retd.), Sri Ramabhadra Junior College, Ramnagar, Hyderabad

Sri Potukuchi Rajamouli, Sr. Lecturer in Mathematics (Retd.), P.B. Siddhardha College of Arts and Science(A), Vijayawada

Subject Committee Members of BIE

Prof. D. Rama Murthy, Professor of Mathematics (Retd.), Osmania University, Hyderabad

Prof. S. Raj Reddy, Professor of Mathematics (Retd.), Kakatiya University, Warangal

Prof. D.R.V. Prasada Rao, Professor of Mathematics (Retd.), S.K.University, Ananthapur

Dr. D. Chitti Babu, Reader. Govt. College (Autonomous), Rajahmundry

Dr. C.T. Suryanarayana Chari, Reader, Silver Jubilee Govt. College. Kurnool

Dr. Y. Bhaskar Reddy, Reader, Govt. Degree College, Rangasaipet. Warangal

Dr. M.V.N. Patrudu, Principal. Govt. Junior College, Veeraghattam, Srikakulam (Dist)

Sri T. Markandeya Naidu, Lecturer. P.V.K.N. Govt. Degree College. Chittoor

Sri C. Linga Reddy, Principal, Govt. Junior College (Boys), Nirmal, Adilabad (Dist)

Sri Kodi Nageswara Rao, School Assistant (Retd.), Chagallu, West Godavari (Dist)

Sri C. Sadasiva Sastry, Lecturer in Mathematics (Retd.), Sri Ramabhadra Jr. College, Hyderabad

Smt. S.V. Sailaja, Junior Lecturer. New Govt. Junior College, Kukatpally, Hyderabad

Sri K. Chandra Sekhara Rao, Junior Lecturer, GJC, Uppugundur. Prakasam (Dist)

Sri P. Harinadha Chary, Junior Lecturer, Sri Srinivasa Jr. College, Tiruchanur

Panel of Experts of BIE

Prof. D. Rama Murthy, Professor of Mathematics (Retd.), Osmania University, Hyderabad

Prof. D.R.V. Prasada Rao, Professor of Mathematics (Retd.), S.K. University. Ananthapur

Dr, C.T. Suryanarayana Chari, Reader in Mathematics, Silver Jubilee Government College (Autonomous) , Kurnool

Sri C. Sadasiva Sastry, Lecturer in Mathematics (Retd.), Sri Ramabhadra Junlor College. Hyderabad

Dr. M.V.N. Patrudu, Principal, Govt. Junior College, Veeraghattam. Srikakulam (Dist)

Smt. S.V. Sailaja, Junior Lecturer, New Govt. Junior College, Kukatpally. Hyderabad

Text Book Review Committee

Editors

Prof. K. Rama Mohan Rao
Professor of Applied Mathematics (Retd.)
Andhra University, Visakhapatnam.

Prof. G. Chakradhara Rao
Professor of Mathematics (Retd.)
Andhra University, Visakhapatnam.

Reviewed by

Prof. S. Kalesha Vali
Department of Engineering Mathematics
Andhra University, Visakhapatnam.

Dr. Y. Purushothama Reddy
Principal (Retd.)
Government Degree Collge, Bantumilli,
Krishna Dist.

Sri K.V.S. Prasad
Lecturer in Mathematics (Retd.)
Government College for Women,
Srikakulam.

Dr. K. Ravi Babu
Lecturer in Mathematics
Dr. V.S. Krishna Govt. Degree & P.G. Collge(A),
Visakhapatnam.

Coordinating Committee of Board of Intermediate Education, A.P.

Sri M.V. Seshagiri Babu, I.A.S.
Secretary
Board of Intermediate Education,
Andhra Pradesh

Educational Research & Training Wing (Text Books)

Dr. A. Srinivasulu
Professor

Sri. M. Ravi Sankar Naik
Assistant Professor

Dr. M. Ramana Reddy
Assistant Professor

Sri J.V. Ramana Gupta
Assistant Professor

Telugu and Sanskrit Akademi, Andhra Pradesh
Coordinating Committee

Sri V. Ramakrishna, I.R.S.
Director

Dr. M. Koteswaramma, M.Com., Ph.D.
Research Officer

Dr. S.A.T. Rajyalakshmi M.Sc., B.Ed., M.A., Ph.D.
Research Assistant

Dr. K. Glory Sathyavani, M.Sc., Ph.D., M.Ed.
Research Assistant

Foreword

The Government of India vowed to remove the educational disparities and adopt a common core curriculum across the country especially at the Intermediate level. Ever since the Government of Andhra Pradesh and the Board of Intermediate Education (BIE) swung into action with the task of evolving a revised syllabus in all the Science subjects on par with that of CBSE, approved by NCERT, its chief intention being enabling the students from Andhra Pradesh to prepare for the National Level Common Entrance tests like NEET, ISEET etc for admission into Institutions of professional courses in our Country.

For the first time BIE AP has decided to prepare the Science textbooks. Accordingly an Academic Review Committee was constituted with the Commissioner of Intermediate Education, AP as Chairman and the Secretary, BIE AP; the Director SCERT and the Director Telugu Akademi as members. The National and State Level Educational luminaries were involved in the textbook preparation, who did it with meticulous care. The textbooks are printed on the lines of NCERT maintaining National Level Standards.

The Education Department of Government of Andhra Pradesh has taken a decision to publish and to supply all the text books with free of cost for the students of all Government and Aided Junior Colleges of newly formed state of Andhra Pradesh.

We express our sincere gratitude to the Director, NCERT for according permission to adopt its syllabi and curriculum of Science textbooks. We have been permitted to make use of their textbooks which will be of great advantage to our student community. I also express my gratitude to the Chairman, BIE and the honorable Minister for HRD and Vice Chairman, BIE and Secretary (SE) for their dedicated sincere guidance and help.

I sincerely hope that the assorted methods of innovation that are adopted in the preparation of these textbooks will be of great help and guidance to the students.

I wholeheartedly appreciate the sincere endeavors of the Textbook Development Committee which has accomplished this noble task.

Constructive suggestions are solicited for the improvement of this textbook from the students, teachers and general public in the subjects concerned so that next edition will be revised duly incorporating these suggestions.

It is very much commendable that Intermediate text books are being printed for the first time by the Akademi from the 2021-22 academic year.

Sri. V. Ramakrishna I.R.S.
Director
Telugu and Sanskrit Akademi,
Andhra Pradesh

Preface

The Board of Intermediate Education (AP), has recently revised the syllabus in Mathematics for the Intermediate Course with effect from the Akademic year 2012-13. Accordingly Telugu Akademi has prepared the necessary Text Books in Mathematics.

In accordance with the current syllabus, the topics relating to paper I-A : **Algebra, Vector Algebra** and **Trigonometry** are dealt with in this book. The syllabus is presented in ten chapters. Algebra part given in three chapters : **Functions, Mathematical Induction** and **Matrices** Vector Algebra part given in two chapters : **Addition of Vectors** and **Product of Vectors.** Trigonometry part given in five chapters : **Trigonometric Ratios upto Transformation, Trigonomitric Equations, Inverse Trigonometric Functions, Hyperbolic Functions** and **Properties of Triangles.**

Further, for the benfit of students intending to appear for All India Level Competitive Examinations, the Additional Reading Material is included in the Appendix, It contains four chapters : **Sets, Relations, Sequences and Series** and **Mathematical Reasoning**. These topics are for additional reading, but not for examinations. **No question will be set on the Additional Reading Material, in the Intermediate I Year Public Examination, Mathematics, paper- IA.**

Every chapter herein. is divided into various sections and subsections, depending on the contents discussed. These contents are strictly in accordance with the prescribed syllabus and they reflect faithfully, the scope and spirit of the same. Necessary definitions, theorems, Corollaries, proofs and notes are given in detail. Key concepts are given at the end of each chapter. Illustrative examples and solved problems are in plenty, and these shall help the students in understanding the subject matter.

Every chapter contains exercises in a graded manner which enable the students to solve them by applying the knowledge acquired. All these problems are classified according to the nature of their answers as **I - very short II short and III-long.** Answers are provided for all the exercises at the end of each chapter.

Keeping in view the National level competitive examinations, some concepts and notions are highlighled for the benefit of the students. Care has been taken regarding rigor and logical conSistency in the presentation of concepts and in proving theorems. At the end of the text Book, a lisl of some **Reference Books** in the subject malter is furnished.

The Members of the Mathematics Subject Committee, constituted by Board of Intermediate Education, were invited to interact with the team of the Authors and Editors. They pursued the contents chapter wise and gave some useful suggestions and comments which are duly incorporated. The special feature of this Book, brought out in a new format, is that each chapter begins with a thought mostly on Mathematics. through a quotation from a famous thinker. It carries a portrait of a noted mathematician with a brief write-up.

In the concluding part of each chapter some relevant historical notes are appended. Wherever found appropriate, references are also made of the contributions of ancient Indian scientists to the advancement of Methamatics. The purpose is to enable the students to have a glimpse into the history of Mathematics in general and the contributions of Indian mathematicians in particular.

Inspite of enough care taken in the scrutiny at various stages in the preparation of the book, errors might have crept in. The readers are therefore, requested to identify and bring them to the notice of the Akademi. We will appreciate if suggetions to enhance the quality of the book are given. Efforts will be made to incorporate them in the subsequent editions.

Prof. P.V. Arunachalam
Chief Coordinator

Preface to the Reviewed Edition

Telugu Akademi is publishing Text books for Two year Intermediate in English and Telugu medium since its inception, periodical review and revision of these publications has been undertaken as and when there was an updation of Intermediate syllabus.

In this reviewed Edition, now being undertaken by the Telugu Akademi, Andhra Pradesh the basic content of its earlier Edition is considered and it is reviewed by a team of experienced teachers. Modification by way of correcting errors, print mistakes, incorporating additional content where necessary to elucidate a concept and / or a definition, modification of existing content to remove obscurities for releasing the concept more easily are carried out mainly in this review.

Not withstanding the effort and time spent by the review team in this endeavour, still a few aspects that still need modification or change might have been left unnoticed.

Constructive suggestions from the academic fraternity are welcome and the Akademi will take necessary steps to incorporate them in the forth coming edition.

We appreciate the encouragement and support extended by the Academic and Administrative staff of the Telugu Akademi in fulfilling our assignment with satisfaction.

Editors
(Reviewed Edition)

Contents

1. Functions 1 - 38

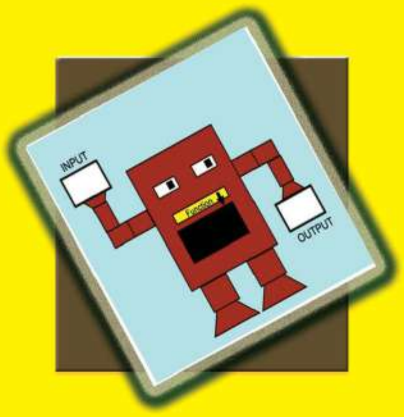
2. Mathematical Induction 39 - 54

3. Matrices 55 - 130

Appendix

(No Question is to be set in the IPE, Mathematics - IA from the topics mentioned below)

Algebra



Chapter 1

Functions

*"The word **function** (or its Latin equivalent) seems to have been introduced into mathematics by Leibnitz in 1694. The concept now dominates much of mathematics and is indispensable in sciences"*

– E.T. Bell

Introduction

All the scientists use mathematics essentially to study relationships. Physicists, Chemists, Engineers, Biologists and Social Scientists, all seek to discern connection among the various elements of their chosen fields and so to arrive at a clear understanding of why these elements behave the way they do. A **function** is a special case of a relation.

The famous mathematician ***Lejeune Dirichlet*** (1805 - 1859) defined a function as follows. A variable is a symbol which represents any one of a set of numbers; if two variables x and y are so related that whenever a value is assigned to x there is automatically assigned, by some rule or correspondence, a value to y, then we say y is a (single valued) function of x, the permissible values that x may assume constitute the domain of definition of the function, and the values taken on by y constitute the range of values of the function.

Dirichlet

(1805 - 1859)

__Johann Peter Gustav Lejeune Dirichlet__ was a German mathematician credited with the modern "formal" definition of a function. He was a student of Gauss. After Gauss's death in 1855, he was appointed as Gauss's successor at Gottingen.

The above definition is a very broad one and does not imply anything regarding the possibility of expressing the relationship between x and y by some kind of analytic expression. It stresses the basic idea of a relationship between two sets. Set theory has naturally extended the concept of function to embrace relationships between any two sets of elements.

In this chapter we focus our attention on a special type of relation, a function, that plays an important role in mathematics and its many applications. Here we study its basic properties and then discuss several special types of functions. In order to have various important applications of functions later, it is essential to get a good grasp of the concepts in this chapter.

1.0 Ordered pairs

Let A and B be sets. If $a \in A$ and $b \in B$ then (a, b) is an **ordered pair**. 'a' is called the **first component** (coordinate) and 'b' is called **the second component** (coordinate) of the ordered pair (a, b). For example, the coordinates of a point in a plane are ordered pairs of real numbers. If (a_1, b_1) and (a_2, b_2) are ordered pairs, then

$$(a_1, b_1) = (a_2, b_2) \Leftrightarrow a_1 = a_2 \text{ and } b_1 = b_2 .$$

1.0.1 Definition (Cartesian product)

Let A *and* B *be two sets. Then* $\{(a, b) | a \in A,\ b \in B\}$ *is called the Cartesian product of* A *and* B, *and is denoted by* $A \times B$ *(to be read as* A *cross* B).

1.0.2 Examples

$$\begin{aligned}
\text{If}\quad A &= \{1, 2, 3\},\ B = \{x, y\} \text{ then} \\
A \times B &= \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\} \\
B \times A &= \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\} \\
A \times A &= \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\} \\
B \times B &= \{(x, x), (x, y), (y, x), (y, y)\}
\end{aligned}$$

1.0.3 Note

1. If A and B are distinct non-empty sets then $A \times B \neq B \times A$.
2. If one of the sets A and B is empty, then $A \times B$ is also empty.
3. Some particular notations

   $\mathbb{R}$ **or R** : set of all real numbers

   $\mathbf{R}^+$: set of all positive real numbers : $\{x | x \in \mathbf{R}, x > 0\}$

   Q : set of all rational numbers

   $\mathbf{Q}^+$: set of all positive rational numbers

   N : set of all natural numbers

   Z : set of all integers

If $a, b \in \mathbf{R}$, $a \leq b$ then

$$(a, b) = \{x \in \mathbf{R} \mid a < x < b\}$$

$$(a, b] = \{x \in \mathbf{R} \mid a < x \leq b\}$$

$$[a, b) = \{x \in \mathbf{R} \mid a \leq x < b\}$$

$$[a, b] = \{x \in \mathbf{R} \mid a \leq x \leq b\}$$

$$[a, \infty) = \{x \in \mathbf{R} \mid a \leq x\}$$

$$(a, \infty) = \{x \in \mathbf{R} \mid a < x\}$$

$$(-\infty, a) = \{x \in \mathbf{R} \mid x < a\}$$

$$(-\infty, a] = \{x \in \mathbf{R} \mid x \leq a\}$$

1.0.4 Definition (Relation)

If A and B are non-empty sets, then any subset of A × B is called a ***relation*** *from A to B. In particular, any relation from A to A is called a* ***binary relation*** *on A.*

1.0.5 Examples

If $\mathrm{A} = \{1, 2, 3\}$, $\mathrm{B} = \{\alpha, \beta\}$ then

$$\mathrm{A} \times \mathrm{B} = \{(1, \alpha), (1, \beta), (2, \alpha), (2, \beta), (3, \alpha), (3, \beta)\}$$

(i) $f = \{(1, \alpha), (2, \beta), (3, \alpha)\}$ is a relation from A to B.

(ii) $g = \{(1, \alpha), (1, \beta)\}$ is a relation from A to B.

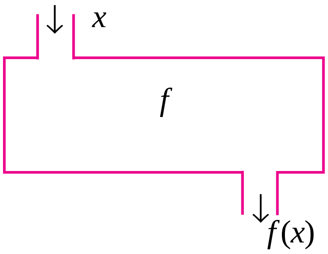
In fact we can define $2^6 = 64$ relations from A to B because the number of elements in $\mathrm{A} \times \mathrm{B}$ is 6 hence there are 2^6 subsets of $\mathrm{A} \times \mathrm{B}$.

1.1 Types of Functions - Definitions

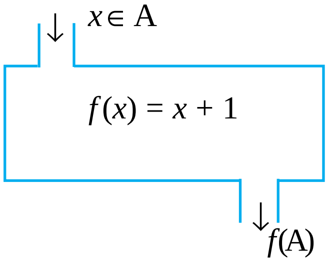
1.1.1 Definition (Function)

Let A and B be non-empty sets and f be a relation from A to B. If for each element $a \in \mathrm{A}$, there exists a unique $b \in \mathrm{B}$ such that $(a, b) \in f$, then f is called a ***function*** *(or mapping) from A to B (or A into B). It is denoted by $f : A \to B$. The set A is called the* ***domain*** *of f and B is called the* ***co-domain*** *of f.*

A function f can also be seen in the following way, which takes an input x and returns an output $f(x)$.



For example, if $f : A \to B$ is a function defined as $f(x) = x + 1$ and $A = \{1, 2, 3\}$, then $f(A) = \{2, 3, 4\}$.



1.1.2 Note

A relation f from A to B (*i.e.* $f \subseteq A \times B$) is a function from A to B if for each $a \in A$, there exists exactly one $b \in B$ such that $(a, b) \in f$ and this 'b' will be denoted by $f(a)$. In other words, for each $a \in A$, there exists a unique element $f(a) \in B$ such that $(a, f(a)) \in f$.

1.1.3 Definition (Image, Pre-Image)

If $f : A \to B$ is a function and if $f(a) = b$, then 'b' is called the ***image*** *of 'a' under f or the f-image of a. The element 'a' is called a* ***pre-image*** *or an* ***inverse image*** *of b under f and is denoted by $f^{-1}(b)$. More generally if $E \subseteq B$, $f^{-1}(E) = \{x \mid x \in A,\ f(x) \in E\}$ is called the inverse image of E under f. Then $f^{-1}(b) = f^{-1}(\{b\})$ if $b \in B$.*

1.1.4 Examples

1. Example: The relation $f = \{(x, x^2 + 1) \mid x \in \mathbf{R}\}$ is a function from $\mathbf{R}$ to $\mathbf{R}^+$, since every $x \in \mathbf{R}$ has association with unique element $x^2 + 1$ in $\mathbf{R}^+$. The function $f : \mathbf{R} \to \mathbf{R}^+$ is given by $f(x) = x^2 + 1$. Observe that $f(1) = 2$ and $f(-1) = 2$. Note that the numbers less than 1 have no pre-image under f.

2. Example: The relation $f = \left\{(x, \frac{1}{x}) \mid 0 \neq x \in \mathbf{R}\right\}$ is not a function from $\mathbf{R}$ to $\mathbf{R}$ since there is no b in $\mathbf{R}$ such that $(0, b) \in f$. But $f(x) = \frac{1}{x}$ is a function from $\mathbf{R} \setminus \{0\} \to \mathbf{R}$ since every $x \in \mathbf{R} \setminus \{0\}$ has association with a unique element in $\mathbf{R}$.

1.1.5 Definition (Range)

If $f : A \to B$ is a function, then $f(A)$, the set of all f- images of elements in A, is called the ***range*** *of f. Clearly $f(A) = \{f(a) \mid a \in A\} \subseteq B$.*

Also $f(A) = \{b \in B \mid b = f(a)$ for some $a \in A\}$.

1.1.6 Examples

1. Example: Let $f : \mathbf{N} \to \mathbf{N}$ be defined by $f(n) = 2n$.

Then the range of $f = f(\mathbf{N}) = \{2n \mid n \in \mathbf{N}\}$ = set of all even natural numbers.

2. Example: Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2$.

Then the range of $f = f(\mathbf{R}) = \{x^2 \mid x \in \mathbf{R}\} = [0, \infty)$ $[\because x^2 \geq 0$ for all $x \in \mathbf{R}]$.

1.1.7 Definition (Injection or one - one function)

A function $f : A \to B$ is called an ***injection*** *if distinct elements of A have distinct f- images in B. An injection is also called a* ***one-one function****.*

$f : A \to B$ is an injection $\Leftrightarrow a_1, a_2 \in A$ and $a_1 \neq a_2$ implies that $f(a_1) \neq f(a_2)$

$\Leftrightarrow a_1, a_2 \in A$ and $f(a_1) = f(a_2)$ implies that $a_1 = a_2$

1.1.8 Examples

1. Example

Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4, 5\}$

(i) If $f = \{(a, 3), (b, 5), (c, 1), (d, 4)\}$ then f is a function from A into B and for different elements in A, there are different f- images in B. Hence f is an injection.

(ii) If $g = \{(a, 2), (b, 2), (c, 3), (d, 5)\}$, then g is a function from A into B, but $g(a) = g(b)$. Hence 'g' is not an injection.

2. Example

Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x + 1$. Then 'f' is an injection since for any $a_1, a_2 \in \mathbf{R}$ and $f(a_1) = f(a_2) \Rightarrow 2a_1 + 1 = 2a_2 + 1 \Rightarrow a_1 = a_2$.

3. Example

Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2$. Then 'f' is not an injection because $f(-1) = 1 = f(1)$.

4. Example

Let $f : \mathbf{N} \to \mathbf{N}$ be defined by $f(x) = x^2$. Then 'f' is an injection since for $a_1, a_2 \in \mathbf{N}$ and

$$f(a_1) = f(a_2) \Rightarrow a_1^2 = a_2^2 \Rightarrow (a_1^2 - a_2^2) = 0 \Rightarrow (a_1 - a_2)(a_1 + a_2) = 0$$

$$\Rightarrow a_1 - a_2 = 0 \; [\because a_1, a_2 \in \mathbf{N} \Rightarrow a_1 + a_2 > 0] \Rightarrow a_1 = a_2.$$

5. Example

Let $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$. We can't define an injection from A to B because atleast two distinct elements in A have the same f - image in B for any function $f : A \to B$.

1.1.9 Definition (Surjection)

*A function $f : A \to B$ is called a **surjection** if the range of f is equal to the co-domain of f.*

$f : A \to B$ is a surjection	$\Leftrightarrow$ range $f = f(A) = B$ (*co-domain*)
	$\Leftrightarrow B = \{f(a) \mid a \in A\}$
	$\Leftrightarrow$ for every $b \in B$ there exists atleast one $a \in A$ such that $f(a) = b$

Hence we may conclude that $f : A \to B$ is a surjection if every element of B occurs as the image of atleast one element of A (i.e., every element in B has a 'pre image' in A). A surjection is also called an **onto function**.

1.1.10 Examples

1. Example

Let A = {1, 2, 3, 4} and B = {a, b, c}

(i) If f = {(1, a), (2, b), (3, c), (4, c)} then

f is a function from A to B and range

$f = f(A) = \{a, b, c\} = B$, the

co-domain. Hence it is a surjection.

Note that f is not an injection.

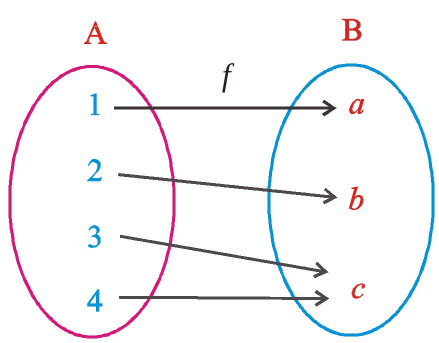


Fig. 1.1

(ii) If $g = \{(1, b), (2, b), (3, c), (4, c)\}$ then g is a function from A to B but not a surjection because there is no pre image to the element $a \in$ B. Note that g is not an injection.

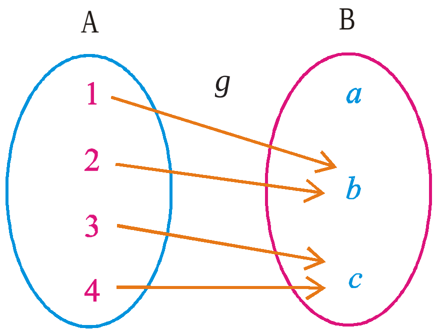


Fig. 1.2

2. Example

Let A = {-3, -2, -1, 1, 2, 3} and B = {1, 4, 9}. If $f : A \to B$ defined by $f(x) = x^2$ for all $x \in A$ then, range $f = f(A) = \{f(-3), f(-2), f(-1), f(1), f(2), f(3)\} = \{1, 4, 9\} = B$.

$\therefore$ $f : A \to B$ is a surjection. Note that f is not an injection.

3. Example

Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = ax + b$ $(a, b \in \mathbf{R}$ and $a \neq 0)$. Then f is a surjection since for any $y \in \mathbf{R}$ (co-domain) there exists $x = \dfrac{y-b}{a} \in \mathbf{R}$ (domain) such that $f(x) = ax + b = \dfrac{a(y-b)}{a} + b = y$ (i.e., every element in the co-domain has a pre-image in the domain). Note that f is an injection too.

4. Example

Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2 + 4$. Then range of $f = [4, \infty)$ [$\because$ for any $x \in \mathbf{R}$ we have $x^2 \geq 0 \Rightarrow x^2 + 4 \geq 4$] and it is not equal to co-domain $\mathbf{R}$. Hence $f : \mathbf{R} \to \mathbf{R}$ is not a surjection. In particular there are no pre-images for all real numbers less than 4 in its co-domain $\mathbf{R}$. Note that f is not an injection.

1.1.11 Definition (Bijection)

*If $f : A \to B$ is both an injection and a surjection then f is said to be a **bijection** or **one-to-one** from A onto B.*

i.e., $f : A \to B$ is a bijection $\Leftrightarrow$ f is both injection and surjection

$\Leftrightarrow$ (i) *If $a_1, a_2 \in A$ and $f(a_1) = f(a_2)$ then $a_1 = a_2$*

(ii) *for every $b \in B$ there exists atleast one $a \in A$ such that $f(a) = b$.*

1.1.12 Definition (Finite set)

If A is empty or there exists $n \in \mathbf{N}$ such that there is a bijection from A onto $\{1, 2, 3, ..., n\}$ then A is called a ***finite set****. In such a case we say that the number of elements in A is n and denote it by $|A|$ or $n(A)$.*

1.1.13 Remarks

(i) In particular, if A and B are two finite sets with $|A| > |B|$ then we can't define an injection from A into B. Hence if there is an injection from A to B then $|A| \le |B|$. The converse of this also holds good, that is, if A and B are finite sets such that $|A| \le |B|$, then we can define an injection $f : A \to B$, for, if $A = \{a_1, a_2, ..., a_n\}$ then there exist distinct elements $b_1, b_2, ..., b_n \in B$ (since $n = |A| \le |B|$) and the function $f : A \to B$, defined by $f(a_i) = b_i$, for $1 \le i \le n$, is an injection.

(ii) Let A and B be two finite sets and $|A| < |B|$, then we can't define a surjection from A to B. Since if $f : A \to B$ then range $f = f(A)$ contains atmost $|A|$ elements $\ne |B|$ (codomain) $[\because |A| < |B|]$. Hence if there is an onto function from A to B then $|A| \ge |B|$. The converse of this also holds good. That is if A and B are finite sets such that $|A| \ge |B|$, then we can define a surjection $f : A \to B$; for if $B = \{b_1, b_2, ..., b_n\}$ then $n \le |A|$ and hence there exist distinct elements $a_1, a_2, ..., a_n \in A$ and we can define $f : A \to B$ by

$$f(a) = \begin{cases} b_i & \text{if } a = a_i \text{ for some } i \\ b_1 & \text{if } a \ne a_i \text{ for all } i \end{cases}$$, which becomes a surjection.

(iii) Note that if there is a bijection 'f' from a finite set A to a finite set B then, since f is both injection and surjection, $|A| \le |B|$ and $|A| \ge |B|$ hence $|A| = |B|$. Thus for any two finite sets A and B, $|A| = |B|$ if and only if there is a bijection $f : A \to B$.

1.1.14 Example

Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x + 3$, then from example 3 (1.1.10), f is a bijection. However if we change the domain of f as **N** then $f(x) = 2x + 3 \in \mathbf{N}$ $\forall x \in \mathbf{N}$. Also,

(i) If $x_1, x_2 \in \mathbf{N}$ (domain), $f(x_1) = f(x_2) \Rightarrow 2x_1 + 3 = 2x_2 + 3 \Rightarrow x_1 = x_2$.

$\therefore f : \mathbf{N} \to \mathbf{N}$ is an injection.

(ii) Range $f = f(\mathbf{N}) = \{f(1), f(2), f(3), ...\} = \{5,7,9,...\} \neq \mathbf{N}$. (codomain)

Hence $f : \mathbf{N} \to \mathbf{N}$ is not a surjection.

Observe that the natural numbers less than 5 and the even natural numbers in the co-domain $\mathbf{N}$ of f have no pre-images in domain $\mathbf{N}$.

1.1.15 Definition (Equality of functions)

Let f and g be functions. We say f and g are ***equal*** *and write $f = g$ if domain of f = domain of g and $f(x) = g(x)$ for all $x \in$ domain f.*

Problem : *On what domain the functions $f(x) = x^2 - 2x$ and $g(x) = -x + 6$ are equal?*

Solution : $f(x) = g(x)$

$\Leftrightarrow \quad x^2 - 2x = -x + 6$

$\Leftrightarrow \quad x^2 - x - 6 = 0$

$\Leftrightarrow \quad (x-3)(x+2) = 0$

$\Leftrightarrow \quad x = -2, 3$

$\therefore$ $f(x)$ and $g(x)$ are equal on the domain $\{-2, 3\}$.

1.1.16 Definition (Constant function)

A function $f : A \to B$ is said to be a ***constant function****, if the range of f contains one and only one element. i.e. $f(x) = c$ for all $x \in A$, for some fixed $c \in B$. In this case the constant function f will be denoted by 'c' itself.*

1.1.17 Example

Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$ and $f = \{(a, 2), (b, 2), (c, 2), (d, 2)\}$ then $f : A \to B$ is a constant function.

1.1.18 Definition (Identity function)

Let A be a non-empty set. Then the function $f : A \to A$ defined by $f(x) = x$ for all $x \in A$ is called the ***identity function*** *on A and is denoted by I_A.*

1.1.19 Example

If $A = \{a, b, c\}$, then $I_A = \{(a, a), (b, b), (c, c)\}$.

1.1.20 Solved Problems

1. Problem: *If $f : \mathbf{R} \setminus \{0\} \to \mathbf{R}$ is defined by $f(x) = x + \frac{1}{x}$ then prove that* $(f(x))^2 = f(x^2) + f(1)$.

Solution: Since $f(x) = \left(x + \frac{1}{x}\right)$

$$f(x^2) + f(1) = x^2 + \frac{1}{x^2} + \left(1 + \frac{1}{1}\right) = x^2 + \frac{1}{x^2} + 2$$

$$= \left(x + \frac{1}{x}\right)^2 = (f(x))^2.$$

2. Problem: *If the function f is defined by* $f(x) = \begin{cases} 3x-2, x>3 \\ x^2-2, -2 \le x \le 2 \\ 2x+1, x<-3 \end{cases}$

then find the values, if exist, of $f(4)$, $f(2.5)$, $f(-2)$, $f(-4)$, $f(0)$, $f(-7)$.

Solution: Note that the domain of f is $(-\infty,-3) \cup [-2,2] \cup (3,\infty)$.

(i) Since $f(x) = 3x-2$, for $x > 3$, we have $f(4) = 12-2 = 10$

(ii) 2.5 does not belong to domain f, $f(2.5)$ is not defined.

(iii) Since $f(x) = x^2-2, -2 \le x \le 2$, we have $f(-2) = (-2)^2 - 2 = 2$

(iv) Since $f(x) = 2x+1, x<-3$, we have $f(-4) = 2(-4)+1 = -7$

(v) Since $f(x) = x^2 - 2$ when $-2 \le x \le 2$, we have $f(0) = 0^2 - 2 = -2$

(vi) Since $f(x) = 2x+1$, for $x < -3$, we have $f(-7) = 2(-7)+1 = -13$.

3. Problem: *If* $A = \left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\right\}$ *and* $f : A \to B$ *is a surjection defined by* $f(x) = \cos x$ *then find* B.

Solution: Let $f : A \to B$ be a surjection defined by $f(x) = \cos x$.

Then $B = \text{range } f = f(A) = \left\{f(0), f\left(\frac{\pi}{6}\right), f\left(\frac{\pi}{4}\right), f\left(\frac{\pi}{3}\right), f\left(\frac{\pi}{2}\right)\right\}$

$$= \left\{\cos 0, \cos\frac{\pi}{6}, \cos\frac{\pi}{4}, \cos\frac{\pi}{3}, \cos\frac{\pi}{2}\right\}$$

$$= \left\{1, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}, 0\right\}.$$

4. Problem: *Determine whether the function $f : \mathbf{R} \to \mathbf{R}$ defined by*

$$f(x) = \frac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$$ *is an injection or a surjection or a bijection.*

Solution: Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \frac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$, then f is not an injection as

$f\ (0) = \dfrac{e^0 - e^0}{e^0 + e^0} = 0$ and $f\ (-1) = \dfrac{e - e}{e^{-1} + e} = 0$ and also f is not a surjection since, for $y = 1$ there is no $x \in \mathbf{R}$ such that $f\ (x) = 1$.

If there is such $x \in \mathbf{R}$ then $e^{|x|} - e^{-x} = e^x + e^{-x}$, clearly $x \neq 0$

for $x > 0$ this equation gives $-e^{-x} = e^{-x}$ which is not possible

for $x < 0$ this equation gives $-e^{-x} = e^x$ which is also not possible.

5. Problem: *Determine whether the function* $f\ :\ \mathbf{R} \to \mathbf{R}$ *defined by*

$f\ (x) = \begin{cases} x & if \quad x > 2 \\ 5x - 2 & if \quad x \leq 2 \end{cases}$ *is an injection or a surjection or a bijection.*

Solution: Since 3 > 2, we have $f(3) = 3$,

Since 1 < 2, we have $f(1) = 5(1) - 2 = 3$

$\therefore$ 1 and 3 have same 'f' image. Hence f is not an injection.

Let $y \in \mathbf{R}$ then $y > 2$ (or) $y \leq 2$

If $y > 2$ take $x = y \in \mathbf{R}$ so that $f\ (x) = x = y$.

If $y \leq 2$ take $x = \dfrac{y+2}{5} \in \mathbf{R}$ and $x = \dfrac{y+2}{5} < 1$.

$\therefore\ f\ (x) = 5x - 2 = 5\left(\dfrac{y+2}{5}\right) - 2 = y$.

$\therefore\ f$ is a surjection.

Since f is not an injection, it is not a bijection.

6. Problem: *Find the domain of definition of the function y(x), given by the equation* $2^x + 2^y = 2$.

Solution: $2^x = 2 - 2^y < 2 \qquad (\because\ 2^y > 0)$

$\Rightarrow \log_2 2^x < \log_2 2$

$\Rightarrow \quad x < 1$

$\therefore$ Domain $= (-\infty, 1)$.

7. Problem: *If* $f : \mathbf{R} \to \mathbf{R}$ *is defined as* $f(x + y) = f(x) + f(y)\ \ \forall\, x, y \in \mathbf{R}$ *and* $f\ (1) = 7$, *then find* $\sum_{r=1}^{n} f\ (r)$.

Solution: Consider $f(2) = f(1 + 1) = f(1) + f(1) = 2\, f(1)$

$f(3) = f(2 + 1) = f(2) + f(1) = 3\, f(1)$

Similarly $f(r) = r\, f(1)$

$$\therefore \sum_{r=1}^{n} f\ (r) = f\ (1) + f\ (2) + \ldots.. + f\ (n)$$

$$= f\ (1) + 2\, f\ (1) + \ldots.. + n\, f\ (1)$$

$$= f(1)\,(1 + 2 + ... + n)$$

$$= \frac{7n(n+1)}{2}.$$

8. Problem : *If* $f(x) = \dfrac{\cos^2 x + \sin^4 x}{\sin^2 x + \cos^4 x}\ \forall\ x \in \mathbf{R}$ then show that $f(2012) = 1$.

Solution: $f(x) = \dfrac{\cos^2 x + \sin^4 x}{\sin^2 x + \cos^4 x}$

$$= \frac{1 - \sin^2 x + \sin^4 x}{1 - \cos^2 x + \cos^4 x}$$

$$= \frac{1 - \sin^2 x(1 - \sin^2 x)}{1 - \cos^2 x(1 - \cos^2 x)}$$

$$= \frac{1 - \sin^2 x \cos^2 x}{1 - \sin^2 x \cos^2 x}$$

$$= 1.$$

$\therefore\ f(2012) = 1.$

Exercise 1(a)

I. 1. If the function f is defined by $f(x) = \begin{cases} x+2, & x > 1 \\ 2, & -1 \le x \le 1 \\ x-1, & -3 < x < -1 \end{cases}$, then find the values of

(i) $f(3)$, (ii) $f(0)$, (iii) $f(-1.5)$,

(iv) $f(2) + f(-2)$, (v) $f(-5)$

2. If $f : \mathbf{R} \setminus \{0\} \to \mathbf{R}$ is defined by $f(x) = x^3 - \dfrac{1}{x^3}$, then show that $f(x) + f(1/x) = 0$.

3. If $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = \dfrac{1-x^2}{1+x^2}$, then show that $f(\tan\theta) = \cos 2\theta$.

4. If $f : \mathbf{R} \setminus \{\pm 1\} \to \mathbf{R}$ is defined by $f(x) = \log\left|\dfrac{1+x}{1-x}\right|$, then show that $f\left(\dfrac{2x}{1+x^2}\right) = 2f(x)$.

5. If $A = \{-2, -1, 0, 1, 2\}$ and $f : A \to B$ is a surjection defined by $f(x) = x^2 + x + 1$, then find B.

6. If $A = \{1, 2, 3, 4\}$ and $f : A \to \mathbf{R}$ is a function defined by

$f(x) = \dfrac{x^2 - x + 1}{x + 1}$, then find the range of f.

7. If $f(x + y) = f(xy)$ $\forall\, x, y \in \mathbf{R}$ then prove that f is a constant function.

II. 1. If $A = \{x \mid -1 \le x \le 1\}$, $f(x) = x^2$, $g(x) = x^3$, which of the following are surjections?

(i) $f : A \to A$ (ii) $g : A \to A$

2. Which of the following are injections or surjections or bijections? Justify your answers.

(i) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \dfrac{2x+1}{3}$

(ii) $f : \mathbf{R} \to (0, \infty)$ defined by $f(x) = 2^x$

(iii) $f : (0, \infty) \to \mathbf{R}$ defined by $f(x) = \log_e x$

(iv) $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = x^2$

(v) $f : \mathbf{R} \to [0, \infty)$ defined by $f(x) = x^2$

(vi) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$

3. Is $g = \{(1,1), (2,3), (3,5), (4,7)\}$ a function from $A = \{1, 2, 3, 4\}$ to $B = \{1, 3, 5, 7\}$? If this is given by the formula $g(x) = ax + b$, then find a and b.

4. If the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \dfrac{3^x + 3^{-x}}{2}$, then show that $f(x+y) + f(x-y) = 2f(x)f(y)$.

5. If the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \dfrac{4^x}{4^x + 2}$, then show that $f(1-x) = 1 - f(x)$, and hence deduce the value of $f\left(\dfrac{1}{4}\right) + 2f\left(\dfrac{1}{2}\right) + f\left(\dfrac{3}{4}\right)$.

6. If the function $f : \{-1, 1\} \to \{0, 2\}$, defined by $f(x) = ax + b$ is a surjection, then find a and b.

7. If $f(x) = \cos(\log x)$, then show that $f\left(\dfrac{1}{x}\right) f\left(\dfrac{1}{y}\right) - \dfrac{1}{2}\left(f\left(\dfrac{x}{y}\right) + f(xy)\right) = 0$.

1.2 Inverse Functions and Theorems

If f is a relation from A to B, then the relation $\{(b, a) \mid (a, b) \in f\}$ is denoted by f^{-1}.

1.2.1 Theorem

If $f : A \to B$ is an injection, then f^{-1} is a bijection from $f(A)$ to A.

Proof: Let $f : A \to B$ be an injection. clearly f^{-1} is a relation from $f(A)$ to A.

Let $b \in f(A)$. Then there exists atleast one $a \in A$ such that $f(a) = b$. Since f is an injection 'a' is the only element of A such that $f(a) = b$. Thus given $b \in f(A)$, there is a unique element a in A such that $(a, b) \in f$. Hence given $b \in f(A)$ there is a unique $a \in A$ such that $(b, a) \in f^{-1}$. Hence f^{-1} is a function from $f(A)$ to A, and $f^{-1}(b) = a$ if and only if $f(a) = b$. Clearly f^{-1} is a surjection. If $b_1, b_2 \in f(A)$ and $f^{-1}(b_1) = f^{-1}(b_2) = a$ (*say*) then $b_1 = f(a) = b_2$. Thus f^{-1} is an injection.

$$\therefore f^{-1} : f(A) \to A \text{ is a bijection.}$$

1.2.2 Corollary

If $f : A \to B$ is a bijection, then f^{-1} is a bijection from B to A.

Proof: This is an immediate consequence of Theorem 1.2.1, since $f(A) = B$.

Note : Since $\left(f^{-1}\right)^{-1} = f$, it follows from 1.2.2 that $f^{-1} : B \to A$ is a bijection if and only if $f : A \to B$ is a bijection.

1.2.3 Definition (Inverse function)

*If $f : A \to B$ is a bijection, then the relation $f^{-1} = \{(b, a) \mid (a, b) \in f\}$ is a function from B to A and is called the **inverse** of f.*

1.2.4 Examples

1. Example: If $A = \{1, 2, 3\}$, $B = \{a, b, c\}$ then $f = \{(1, c), (2, b), (3, a)\}$ is a bijection from A to B and $f^{-1} = \{(a, 3), (b, 2), (c, 1)\}$ is a bijection from B to A.

2. Example: If $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$ then $f = \{(1, c), (2, b), (3, a)\}$ is an injection but not a surjection, $f^{-1} = \{(c, 1), (b, 2), (a, 3)\}$ is a relation from B to A but not a function because $'d' \in B$ has no f^{-1} image in A.

3. Example: If $A = \{1, 2, 3\}$, $B = \{a, b\}$; then $f = \{(1, a), (2, b), (3, a)\}$ is a surjection but not an injection, $f^{-1} = \{(a, 1), (b, 2), (a, 3)\}$ is a relation from B to A but not a function from B to A because for $a \in B$ there are two f^{-1} images in A.

1.2.5 Definition Composite Function

If $f : A \to B$, $g : B \to C$, then the relation $\{(a, g(f(a))) | a \in A\}$ is called ***composite*** *of 'g' with 'f' and is denoted as gof.*

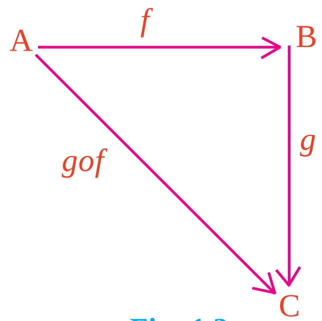


Fig. 1.3

1.2.6 Theorem: *Let $f : A \to B$ and $g : B \to C$ be functions. Then gof is a function from A to C, and (gof) (a) = g (f (a)) for all $a \in A$.*

Proof : Let $a \in A$. Since f is a function from A to B then $f(a) \in B$. Since g is a function from B to C then $g(f(a)) \in C$. Hence *gof* is a relation from A to C. Further, given $a \in A$ there is one and only one element c in C, namely, $g(f(a))$, such that $(a, c) \in gof$. Hence *gof* is a function from A to C and $(gof)(a) = g(f(a))$ for all $a \in A$.

1.2.7 Theorem: *Let $f : A \to B$, $g : B \to C$ be injections, then $gof : A \to C$ is an injection.*

Proof : Let $a_1, a_2 \in A$ be such that $(gof)(a_1) = (gof)(a_2)$

$$g(f(a_1)) = g(f(a_2))$$

$$f(a_1) = f(a_2) \text{ [since } g \text{ is an injection]}$$

$$a_1 = a_2 \quad \text{[since } f \text{ is an injection]}$$

$\therefore gof : A \to C$ is an injection.

1.2.8 Theorem: *Let $f : A \to B$ and $g : B \to C$ be such that gof is an injection. Then f is an injection.*

Proof : Let $a_1, a_2 \in A$ be such that $f(a_1) = f(a_2)$ then $g(f(a_1)) = g(f(a_2))$

$$\Rightarrow \quad gof(a_1) = gof(a_2)$$

$$\Rightarrow \quad a_1 = a_2. \; [\because gof \text{ is an injection}]$$

$\therefore$ f is an injection.

1.2.9 Note

If $f : A \to B$, $g : B \to C$ are such that *gof* is injection then g need not be injection. For example, let $A = \{1, 2\}$, $B = \{a, b, c\}$, $C = \{d, e\}$, $f = \{(1, a), (2, b)\}$ and $g = \{(a, d), (b, e), (c, e)\}$, then $gof = \{(1, d), (2, e)\}$.

Hence gof is an injection but g is not an injection. However, if gof is an injection then necessarily f is an injection.

1.2.10 Theorem: *Let* $f : A \to B$, $g : B \to C$ *be surjections. Then* $gof : A \to C$ *is a surjection.*

Proof : Let $c \in C$. Since $g : B \to C$ is a surjection then there exists $b \in B$ such that $g(b) = c$. Since $f : A \to B$ is a surjection then there exists $'a' \in A$ such that $f(a) = b$.

$$\therefore c = g(b) = g(f(a)) = (gof)(a).$$

$$\therefore \text{ for each } 'c' \in C \text{ there exists } 'a' \in A \text{ such that } (gof)(a) = c.$$

Hence $gof : A \to C$ is a surjection.

1.2.11 Theorem: *Let* $f : A \to B$, $g : B \to C$ *be such that* gof *is a surjection. Then* g *is a surjection.*

Proof : Let $c \in C$. Since $gof : A \to C$ is a surjection then there exists $a \in A$ such that $(gof)(a) = c$, i.e., $g(f(a)) = c$. Let $b = f(a)$. Then $f(a) = b \in B$ and $g(b) = c$.

$\therefore g$ is a surjection.

1.2.12 Note

If $f : A \to B$, $g : B \to C$ are such that gof is a surjection then f need not be a surjection. In Note 1.2.9, gof is a surjection but f is not a surjection. However, if gof is a surjection then necessarily 'g' is a surjection.

1.2.13 Theorem: *Let* $f : A \to B$, $g : B \to C$ *be bijections. Then* $gof : A \to C$ *is a bijection.*

Proof: This is a consequence of Theorems 1.2.7 and 1.2.10.

1.2.14 Theorem: *Let* $f : A \to B$, $g : B \to C$ *be bijections. Then* $(gof)^{-1} = f^{-1}og^{-1}$.

Proof: Since $f : A \to B$, $g : B \to C$ are bijections, so is gof from A to C (from Theorem 1.2.13). Hence $(gof)^{-1}$ is a bijection from C to A. Further, $f^{-1} : B \to A$; $g^{-1} : C \to B$ are also bijections. Hence $f^{-1}og^{-1}$ is a bijection from C to A.

$\therefore$ The functions $(gof)^{-1}$ and $f^{-1}og^{-1}$ are defined on the same domain 'C'.

Let $c \in C$. Since $g : B \to C$ is a bijection, there exists a unique $b \in B$ such that $g(b) = c$ i.e., $g^{-1}(c) = b$.

Now $b \in B$ and $f : A \to B$ is a bijection. Hence there exists a unique $a \in A$ such that $f(a) = b$ i.e., $f^{-1}(b) = a$.

Thus $c = g\ (b) = g\left(f\ (a)\right) = (gof\ )\ (a)$ (or) $(gof\ )^{-1}(c) = a$

Now $(f^{-1}og^{-1})(c) = f^{-1}(g^{-1}(c)) = f^{-1}(b) = a$

Hence $(gof\ )^{-1} = f^{-1}og^{-1}$.

1.2.15 Theorem: *The identity function $I_A : A \to A$ is a bijection and $I_A^{-1} = I_A$.*

Proof: We have $I_A = \{(a, a) \mid a \in \mathrm{A}\}$.

Given $a \in \mathrm{A}$ we have $I_A(a) = a$. Hence I_A is a surjection.

Let $a_1, a_2 \in \mathrm{A}, I_A(a_1) = I_A(a_2) \Rightarrow a_1 = a_2$. Hence I_A is an injection.

$$\therefore I_A : \mathrm{A} \to \mathrm{A} \text{ is bijection and } I_A^{-1} = I_A.$$

1.2.16 Theorem: *Let $f : \mathrm{A} \to \mathrm{B}$, I_A and I_B be identity functions on A and B respectively. Then $foI_A = f = I_Bof$.*

Proof: Since $I_A : \mathrm{A} \to \mathrm{A}$ and $f : \mathrm{A} \to \mathrm{B}$ are functions, foI_A is a function from A to B. Hence functions foI_A and f are defined on same domain A.

Let $a \in \mathrm{A}$, then $(foI_A)\ (a) = f\left(I_A(a)\right) = f(a)$ [$\because I_A\ (a) = a$ for all $a \in \mathrm{A}$]

$$\therefore foI_A = f \qquad \text{... (1)}$$

Since $f : \mathrm{A} \to \mathrm{B}$, $I_B : \mathrm{B} \to \mathrm{B}$, are functions then I_Bof is a function from A to B.

$\therefore$ The functions I_Bof and f are defined on the same domain A.

Let $a \in \mathrm{A}$, then $(I_Bof)\ (a) = I_B\left(f(a)\right) = f(a)$ [$\because f : \mathrm{A} \to \mathrm{B}$ we have $f(a) \in B$]

$$\therefore I_Bof = f \qquad \text{... (2)}$$

From (1) and (2) we have $foI_A = f = I_Bof$.

1.2.17 Theorem: *Let $f : \mathrm{A} \to \mathrm{B}$ be a bijection. Then $fof^{-1} = I_B$ and $f^{-1}of = I_A$.*

Proof : Since $f : \mathrm{A} \to \mathrm{B}$ is a bijection then $f^{-1} : \mathrm{B} \to \mathrm{A}$ is also a bijection. Hence fof^{-1} is a bijection from B to B and $f^{-1}of$ is a bijection from A to A. We have that I_B is a bijection from B to B and I_A is a bijection from A to A.

$\therefore$ The functions fof^{-1} and I_B are defined on the same domain B.

Let $b \in \mathrm{B}$. Since $f : \mathrm{A} \to \mathrm{B}$ is a bijection then there exists a unique $a \in \mathrm{A}$ such that

$$f\ (a) = b \text{ i.e., } f^{-1}(b) = a.$$

$$\text{Thus } fof^{-1}(b) = f\left(f^{-1}(b)\right) = f(a) = b = I_B(b)$$

$$\therefore fof^{-1} = I_B$$

The functions $f^{-1}of$ and I_A are defined on the same domain.

$$\text{We have } f^{-1}of(a) = f^{-1}\left(f(a)\right) = f^{-1}(b) = a = I_A(a)$$

$$\therefore f^{-1}of = I_A.$$

1.2.18 Theorem: *Let $f : A \to B$ be a function. Then f is a bijection if and only if there exists a function $g : B \to A$ such that $fog = I_B$ and $gof = I_A$ and, in this case, $g = f^{-1}$.*

Proof: Let $f : A \to B$ be a bijection. Then $f^{-1} : B \to A$ is a bijection [from Corollary 1.2.2]. Take $g = f^{-1}$. Then from Theorem 1.2.17, $fog = I_B$ and $gof = I_A$.

Conversely, if there exists a function $g : B \to A$ such that $fog = I_B$ and $gof = I_A$ then $gof = I_A$ is an injection, we get from Theorem 1.2.8 that f is an injection. Also, since $fog = I_B$ is a surjection, we get from Theorem 1.2.11 that f is a surjection.

$\therefore f : A \to B$ is a bijection. Hence $f^{-1} : B \to A$ is a bijection. We also have $g : B \to A$.

$\therefore f^{-1}$ and g are defined on the same domain B.

Let $b \in B$. Since $f : A \to B$ is a bijection then there exists a unique $'a' \in A$ such that $f(a) = b$ or $f^{-1}(b) = a$. Now

$$f^{-1}(b) = a = I_A(a) = (gof)\,(a) = g\left(f(a)\right) = g(b)$$

$$\therefore g = f^{-1}.$$

1.2.19 Theorem: *Let $f : A \to B$, $g : B \to C$ and $h : C \to D$. Then $ho(gof) = (hog)of$, that is, composition of functions is associative.*

Proof: Since $f : A \to B$, $g : B \to C$ and $h : C \to D$,

$$gof : A \to C \text{ and } h : C \to D \quad \Rightarrow ho(gof) : A \to D. \text{ Further}$$

$$f : A \to B \text{ and } hog : B \to D \Rightarrow (hog)\,of : A \to D.$$

Thus $ho(gof)$ and $(hog)of$ have the same domain A. Let 'a' be any element of A. Now

$$[ho\,(gof)]\,(a) = h((gof)\,(a)) = h\left(g\left(f(a)\right)\right) = (hog)\left(f(a)\right) = ((hog)of)\,(a)$$

$$\therefore ho\,(gof) = (hog)\,of.$$

1.2.20 Solved Problems

1. Problem: *If* $f:\mathbf{R}\to\mathbf{R},\ g:\mathbf{R}\to\mathbf{R}$ *are defined by* $f(x)=4x-1$ *and* $g(x)=x^2+2$ *then find*

(i) $(gof)(x)$ (ii) $(gof)\left(\frac{a+1}{4}\right)$

(iii) $fof(x)$ (iv) $go(fof)(0)$.

Solution

(i) $(gof)(x)=g\left(f(x)\right)=g\ (4x-1)=(4x-1)^2+2=16x^2-8x+3$ ... (1)

(ii) from (1) we have $(gof)\left(\frac{a+1}{4}\right)=16\left(\frac{a+1}{4}\right)^2-8\left(\frac{a+1}{4}\right)+3=a^2+2$

(iii) $(fof)(x)=f\left(f(x)\right)=f\ (4x-1)=4\ (4x-1)-1=16x-5$ ... (2)

(iv) from (2) we have $(fof)(0)=0-5=-5$

$$\therefore go(fof)(0)=g\left(fof(0)\right)=g(-5)=25+2=27.$$

2. Problem: *If* $f:[0,3]\to[0,3]$ *is defined by* $f(x)=\begin{cases}1+x, & 0\le x\le 2\\ 3-x, & 2<x\le 3\end{cases}$, *then show that* $f[0,3]\subseteq[0,3]$ *and find fof.*

Solution: $0\le x\le 2 \Rightarrow 1\le 1+x\le 3$ ... (1)

$2<x\le 3 \Rightarrow -3\le -x<-2$

$\Rightarrow 3-3\le 3-x<3-2$

$\Rightarrow 0\le 3-x<1$ ... (2)

From (1) and (2), $f[0,3]\subseteq[0,3]$.

When $0\le x\le 1$ we have

$$(fof)(x)=f\left(f(x)\right)=f(1+x)=1+1+x=2+x.\quad [\because 1\le 1+x\le 2]$$

When $1<x\le 2$ we have

$$fof(x)=f\left(f(x)\right)=f(1+x)=3-(1+x)=2-x.\quad [\because 2<1+x\le 3]$$

When $2<x\le 3$ we have

$$fof(x)=f\left(f(x)\right)=f(3-x)=1+3-x=4-x.\quad [\because 0\le 3-x<1]$$

$$\therefore (fof)(x)=\begin{cases}2+x, & 0\le x<1\\ 2-x, & 1<x\le 2\\ 4-x, & 2<x\le 3\end{cases}$$

3. Problem: *If* $f, g : \mathbf{R} \to \mathbf{R}$ *are defined by* $f(x) = \begin{cases} 0 \text{ if } x \in \mathbf{Q} \\ 1 \text{ if } x \notin \mathbf{Q} \end{cases}$

and $g(x) = \begin{cases} -1 \text{ if } x \in \mathbf{Q} \\ 0 \text{ if } x \notin \mathbf{Q} \end{cases}$ *then find* $(fog)(\pi) + (gof)(e)$.

Solution: $(fog)(\pi) = f(g(\pi)) = f(0) = 0$

$(gof)(e) = g(f(e)) = g(1) = -1$

$\therefore (fog)(\pi) + (gof)(e) = -1.$

4. Problem: *Let* $A = \{1,2,3\}$, $B = \{a,b,c\}$, $C = \{p,q,r\}$. *If* $f : A \to B$, $g : B \to C$ *are defined by* $f = \{(1,a), (2,c), (3,b)\}$, $g = \{(a,q),(b,r),(c,p)\}$ *then show that* $f^{-1}og^{-1} = (gof)^{-1}$.

Solution: Given that $f = \{(1,a), (2,c), (3,b)\}$ and $g = \{(a,q), (b,r), (c,p)\}$ then

$$gof = \{(1,q), (2,p), (3,r)\} \Rightarrow (gof)^{-1} = \{(q,1), (p,2), (r,3)\}.$$

$$g^{-1} = \{(q,a), (r,b), (p,c)\},\ f^{-1} = \{(a,1),(c,2), (b,3)\} \text{ then}$$

$$f^{-1}og^{-1} = \{(q,1), (r,3),(p,2)\}.$$

$$\therefore (gof)^{-1} = f^{-1}og^{-1}.$$

5. Problem: *If* $f : \mathbf{Q} \to \mathbf{Q}$ *is defined by* $f(x) = 5x + 4$ *for all* $x \in \mathbf{Q}$, *show that* f *is a bijection and find* f^{-1}.

Solution: Let $x_1, x_2 \in \mathbf{Q}$, $f(x_1) = f(x_2) \Rightarrow 5x_1 + 4 = 5x_2 + 4 \Rightarrow x_1 = x_2$.

$\therefore f$ is an injection.

Let $y \in \mathbf{Q}$. Then $x = \dfrac{y-4}{5} \in \mathbf{Q}$ and

$$f(x) = f\left(\frac{y-4}{5}\right) = 5\left(\frac{y-4}{5}\right) + 4 = y.$$

$\therefore f$ is a surjection and hence f is a bijection.

$\therefore f^{-1} : \mathbf{Q} \to \mathbf{Q}$ is a bijection.

We have $fof^{-1}(x) = I(x)$

$$f\left(f^{-1}(x)\right) = x$$

$$5f^{-1}(x) + 4 = x$$

$$f^{-1}(x) = \frac{x-4}{5} \text{ for all } x \in \mathbf{Q}.$$

Exercise 1(b)

I. 1. If $f(x) = e^x$ and $g(x) = \log_e x$, then show that $fog = gof$ and find f^{-1} and g^{-1}.

2. If $f(y) = \dfrac{y}{\sqrt{1-y^2}}$, $g(y) = \dfrac{y}{\sqrt{1+y^2}}$ then show that $(fog)(y) = y$.

3. If $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are defined by $f(x) = 2x^2 + 3$ and $g(x) = 3x - 2$, then find

(i) $(fog)(x)$, (ii) $(gof)(x)$, (iii) $fof(0)$, (iv) $go(fof)(3)$.

4. If $f : \mathbf{R} \to \mathbf{R}$, $g : \mathbf{R} \to \mathbf{R}$ are defined by $f(x) = 3x - 1$, $g(x) = x^2 + 1$, then find

(i) $fof(x^2+1)$, (ii) $fog(2)$, (iii) $gof(2a-3)$.

5. If $f(x) = \dfrac{1}{x}$, $g(x) = \sqrt{x}$ for all $x \in (0, \infty)$, then find $(gof)(x)$.

6. If $f(x) = 2x - 1$, $g(x) = \dfrac{x+1}{2}$ for all $x \in \mathbf{R}$, then find $(gof)(x)$.

7. If $f(x) = 2$, $g(x) = x^2$, $h(x) = 2x$ for all $x \in \mathbf{R}$, then find $(fo(goh)(x))$.

8. Find the inverse of the following functions.

(i) $a, b \in \mathbf{R}$, $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = ax + b$ $(a \neq 0)$.

(ii) $f : \mathbf{R} \to (0, \infty)$ defined by $f(x) = 5^x$.

(iii) $f : (0, \infty) \to \mathbf{R}$ defined by $f(x) = \log_2 x$.

9. If $f(x) = 1 + x + x^2 + \dots$ for $|x| < 1$ then show that $f^{-1}(x) = \dfrac{x-1}{x}$.

10. If $f : [1, \infty) \to [1, \infty)$ defined by $f(x) = 2^{x(x-1)}$ then find $f^{-1}(x)$.

II. 1. If $f(x) = \dfrac{x-1}{x+1}$, $x \neq \pm 1$, then verify $fof^{-1}(x) = x$.

2. If $A = \{1, 2, 3\}$, $B = \{\alpha, \beta, \gamma\}$, $C = \{p, q, r\}$ and $f : A \to B$, $g : B \to C$ are defined by

$f = \{(1, \alpha), (2, \gamma), (3, \beta)\}$, $g = \{(\alpha, q), (\beta, r), (\gamma, p)\}$,

then show that f and g are bijective functions and $(gof)^{-1} = f^{-1}og^{-1}$.

3. If $f : \mathbf{R} \to \mathbf{R}$, $g : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3x - 2$, $g(x) = x^2 + 1$, then find

(i) $(gof^{-1})(2)$, (ii) $(gof)(x-1)$.

4. Let $f = \{(1, a), (2, c), (4, d), (3, b)\}$ and $g^{-1} = \{(2, a), (4, b), (1, c), (3, d)\}$,

then show that $(gof)^{-1} = f^{-1}og^{-1}$.

5. Let $f:\mathbf{R}\to\mathbf{R}$, $g:\mathbf{R}\to\mathbf{R}$ be defined by $f(x)=2x-3$, $g(x)=x^3+5$ then find $(fog)^{-1}(x)$.

6. Let $f(x)=x^2$, $g(x)=2^x$. Then solve the equation $(fog)(x)=(gof)(x)$.

7. If $f(x)=\dfrac{x+1}{x-1}$ $(x\neq\pm1)$ then find $(fofof)(x)$ and $(fofofof)(x)$.

1.3 Real valued functions (Domain, Range and Inverse)

If X is any set, $f:X\to\mathbf{R}$ then f is called a **real valued function**. For example let $X=\left\{\begin{bmatrix} a & b \\ c & d\end{bmatrix} \middle|\, a,b,c,d\in\mathbf{R}\right\}$, define $f:X\to\mathbf{R}$ by $f(A)=det\ A$ for all $A\in X$, then f is a real valued function.

In this section a function f is defined through a formula, without mentioning the domain and the range explicitly. In such cases, the domain of f is taken to be the set of all real x for which the formula is meaningful. The range of f is the set $\{f(x)\,|\, x$ is in the domain of $f\}$.

1.3.0 (a) : n^{th} root of a non-negative real number

Let x be a non-negative real number and n be a positive integer. Then there exists a unique non-negative real number y such that $y^n=x$. The proof is beyond the scope of this book. This number y is called the n^{th} root of x and is denoted as $x^{1/n}$ (or) $\sqrt[n]{x}$.

When $n=2$, $\sqrt[2]{x}$ is called the square root of x. $\sqrt[2]{x}$ is written simply as $\sqrt{x}$.

If x is any real number and n is an odd positive integer there exists a unique real number y such that $y^n=x$ so that we write $y=\sqrt[n]{x}$ or $x^{\frac{1}{n}}$.

1.3.0 (b): a^x when $1\neq a>0$ and x is a rational number:

If $1\neq a>0$ and $x=\dfrac{m}{n}$ where m, n are integers and $n>0$ we define $a^x=\left(a^m\right)^{\frac{1}{n}}$.

1.3.1 Examples

1. Example: The domain of the real valued function $f(x)=\sqrt{a^2-x^2}$ $(a>0)$ is $[-a, a]$.

[Since $\sqrt{a^2-x^2}\in\mathbf{R}$, $(a>0)\Leftrightarrow a^2-x^2\geq0\Leftrightarrow x^2\leq a^2\Leftrightarrow |x|\leq a\Leftrightarrow -a\leq x\leq a$].

2. Example: The domain of the real valued function $f(x)=\dfrac{1}{2x+1}$ is $\mathbf{R}\setminus\left\{-\dfrac{1}{2}\right\}$.

$$\left[\text{Since } \frac{1}{2x+1}\in\mathbf{R}\Leftrightarrow 2x+1\neq0\Leftrightarrow x\neq-\frac{1}{2}\right]$$

1.3.2 Algebra of real valued functions

If f and g are real valued functions with domains A and B respectively, then both f and g are defined on $A \cap B$ when $A \cap B \neq \phi$.

(i) Let $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$. Suppose that $A \cap B \neq \phi$, we define $f + g$, $f - g$ and fg on $A \cap B$ as $(f \pm g)(x) = f(x) \pm g(x)$ and $(fg)(x) = f(x)g(x)$.

Let $f : A \to \mathbf{R}$ and c be a constant function defined on A. Then from the above definition $(f + c)(x) = f(x) + c$ and $(cf)(x) = cf(x)$ for all $x \in A$. The function $(-1) f$ is denoted by $-f$.

(ii) Let $E = \{x \in A \cap B \mid g(x) \neq 0\} \neq \phi$. We define $\frac{f}{g}$ on E by

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ for all $x \in E$. Note that if $g(x) = 0$, then $\frac{f(x)}{g(x)}$ is not defined.

(iii) Let $f : A \to \mathbf{R}$ and $n \in \mathbf{N}$. We define $|f|$ and f^n on A by $|f|(x) = |f(x)|$ and $f^n(x) = (f(x))^n$ for all $x \in A$.

(iv) If $E = \{x \in A \mid f(x) \geq 0\} \neq \phi$, then we define $\sqrt{f}$ on E by $\sqrt{f}\,(x) = \sqrt{f(x)}$, for all $x \in E$.

In view of the above, we can conclude that if f, g are defined on their respective domains then

domain $(f \pm g)$ = domain of f $\cap$ domain of g

domain (fg) = domain of f $\cap$ domain of g

domain $\left(\frac{f}{g}\right)$ = domain of f $\cap$ domain of g $\cap$ $\{x : g(x) \neq 0\}$

domain $(\sqrt{f})$ = domain of f $\cap$ $\{x : f(x) \geq 0\}$

1.3.3 Solved Problems

1. Problem: *Find the domains of the following real valued functions.*

(i) $f(x) = \dfrac{1}{6x - x^2 - 5}$ (ii) $f(x) = \dfrac{1}{\sqrt{x^2 - a^2}}\ (a > 0)$

(iii) $f(x) = \sqrt{(x+2)(x-3)}$ (iv) $f(x) = \sqrt{(x-\alpha)(\beta - x)}\,(0 < \alpha < \beta)$

(v) $f(x) = \sqrt{2-x} + \sqrt{1+x}$ (vi) $f(x) = \sqrt{x^2 - 1} + \dfrac{1}{\sqrt{x^2 - 3x + 2}}$

(vii) $f(x) = \dfrac{1}{\sqrt{|x| - x}}$ (viii) $f(x) = \sqrt{|x| - x}$

Solution

(i) $f(x) = \dfrac{1}{6x - x^2 - 5} = \dfrac{1}{(x-1)(5-x)} \in \mathbf{R} \Leftrightarrow (x-1)(5-x) \neq 0$

$\Leftrightarrow x \neq 1, 5$

$\therefore$ Domain of f is $\mathbf{R} \setminus \{1, 5\}$.

(ii) $f(x) = \dfrac{1}{\sqrt{x^2 - a^2}} \in \mathbf{R} \Leftrightarrow x^2 - a^2 > 0$

$\Leftrightarrow (x-a)(x+a) > 0$

$\Leftrightarrow x < -a$ (or) $x > a$

$\Leftrightarrow x \in (-\infty, -a) \cup (a, \infty)$

$\therefore$ Domain of f is $(-\infty, -a) \cup (a, \infty) = \mathbf{R} \setminus [-a, a]$.

(iii) $f(x) = \sqrt{(x+2)(x-3)} \in \mathbf{R} \Leftrightarrow (x+2)(x-3) \geq 0$

$\Leftrightarrow x \leq -2$ or $x \geq 3$

$\Leftrightarrow x \in (-\infty, -2] \cup [3, \infty) = \mathbf{R} \setminus (-2, 3)$

$\therefore$ Domain of f is $(-\infty, -2] \cup [3, \infty) = \mathbf{R} \setminus (-2, 3)$.

(iv) $f(x) = \sqrt{(x-\alpha)(\beta - x)} \in \mathbf{R} \Leftrightarrow (x-\alpha)(\beta - x) \geq 0$

$\Leftrightarrow \alpha \leq x \leq \beta \quad (\because \alpha < \beta)$

$\Leftrightarrow x \in [\alpha, \beta]$

$\therefore$ Domain of f is $[\alpha, \beta]$.

(v) $f(x) = \sqrt{2-x} + \sqrt{1+x} \in \mathbf{R} \Leftrightarrow 2 - x \geq 0$ and $1 + x \geq 0$

$\Leftrightarrow 2 \geq x$ and $x \geq -1$

$\Leftrightarrow -1 \leq x \leq 2$

$\Leftrightarrow x \in [-1, 2]$

$\therefore$ Domain of f is $[-1, 2]$.

(vi) $f(x)=\sqrt{x^2-1}+\frac{1}{\sqrt{x^2-3x+2}}\in \mathbf{R} \Leftrightarrow x^2-1\geq 0$ and $x^2-3x+2>0$

$\Leftrightarrow (x+1)(x-1)\geq 0$ and $(x-1)(x-2)>0$

$\Leftrightarrow x\in(-\infty,-1]\cup[1,\infty)$ and $x\in(-\infty,1)\cup(2,\infty)$.

$\Leftrightarrow x\in(\mathbf{R}\setminus(-1,1)\cap(\mathbf{R}\setminus[1,2]))$.

$\Leftrightarrow x\in\mathbf{R}\setminus\{(-1,1)\cup[1,2]\}$

$\Leftrightarrow x\in\mathbf{R}\setminus(-1,2]=(-\infty,-1]\cup(2,\infty)$.

$\therefore$ Domain of f is $(-\infty,-1]\cup(2,\infty)=\mathbf{R}\setminus(-1,2]$.

(vii) $f(x)=\frac{1}{\sqrt{|x|-x}}\in\mathbf{R} \Leftrightarrow |x|-x>0 \Leftrightarrow |x|>x$

$\Leftrightarrow x\in(-\infty,0)$.

$\therefore$ Domain of f is $(-\infty,0)$.

(viii) $f(x)=\sqrt{|x|-x}\in\mathbf{R} \Leftrightarrow |x|-x\geq 0$, which is true for all $x\in\mathbf{R}$.

$\therefore$ Domain of f is $\mathbf{R}$.

2. Problem: *If* $f=\{(4,5),(5,6),(6,-4)\}$ *and* $g=\{(4,-4),(6,5),(8,5)\}$ *then find*

(i) $f+g$ (ii) $f-g$ (iii) $2f+4g$ (iv) $f+4$

(v) fg (vi) f/g (vii) $|f|$ (viii) $\sqrt{f}$

(ix) f^2 (x) f^3

Solution: Domain of $f=A=\{4,5,6\}$, Domain of $g=B=\{4,6,8\}$.

Domain of $f\pm g=A\cap B=\{4,6\}$.

(i) $f+g=\{(4,5-4),(6,-4+5)\}=\{(4,1),(6,1)\}$ and

(ii) $f-g=\{(4,5+4),(6,-4-5)\}=\{(4,9),(6,-9)\}$.

(iii) Domain of $2f=A=\{4,5,6\}$, Domain of $4g=B=\{4,6,8\}$.

$\therefore 2f=\{(4,10),(5,12),(6,-8)\}$, $4g=\{(4,-16),(6,20),(8,20)\}$.

Domain of $2f+4g=\{4,6\}$

$\therefore 2f+4g=\{(4,10-16),(6,-8+20)\}=\{(4,-6),(6,12)\}$.

(iv) Domain of $f+4 = A = \{4,5,6\}$

$$f+4 = \{(4,5+4),(5,6+4),(6,-4+4)\} = \{(4,9),(5,10),(6,0)\}.$$

(v) Domain of $fg = A \cap B = \{4,6\}$

$$fg = \{(4,(5)\ (-4)),(6,(-4)(-5))\} = \{(4,-20),\ (6,20)\}.$$

(vi) Domain of $\frac{f}{g} = \{4,6\}$.

$$\therefore \frac{f}{g} = \left\{\left(4,\ \frac{-5}{4}\right),\left(6,\frac{-4}{5}\right)\right\}.$$

(vii) Domain of $|f| = A = \{4,5,6\}$.

$$\therefore |f| = \{(4,5),\ (5,6),\ (6,4)\}.$$

(viii) Domain of $\sqrt{f} = \{4,5\}$.

$$\therefore \sqrt{f} = \{(4,\sqrt{5}),\ (5,\sqrt{6})\}.$$

(ix) Domain of $f^2 = A = \{4,5,6\}$.

$$\therefore\ f^2 = \{(4,25),(5,36),(6,16)\}.$$

(x) Domain of $f^3 = A = \{4,5,6\}$.

$$\therefore\ f^3 = \{(4,125),\ (5,216),\ (6,-64)\}.$$

3. Problem: *Find the domains and ranges of the following real valued functions.*

(i) $f\ (x) = \frac{2+x}{2-x}$ (ii) $f\ (x) = \frac{x}{1+x^2}$ (iii) $f\ (x) = \sqrt{9-x^2}$

Solution

(i) $\frac{2+x}{2-x} \in \mathbf{R} \Leftrightarrow 2-x \neq 0 \Leftrightarrow x \neq 2 \Leftrightarrow x \in \mathbf{R} \setminus \{2\}$.

$\therefore$ Domain of f is $\mathbf{R} \setminus \{2\}$.

Let $f\ (x) = y \Rightarrow \frac{2+x}{2-x} = y \Rightarrow x = \frac{2\ (y-1)}{(y+1)}$, clearly, x is not defined for

$y+1=0$ i.e., when $y=-1$. $\therefore$ range of $f = \mathbf{R} \setminus \{-1\}$.

(ii) $f\ (x) = \frac{x}{1+x^2} \in \mathbf{R} \Leftrightarrow x \in \mathbf{R}$ is defined for all $x \in \mathbf{R}$, since $x^2 + 1 \neq 0$ for $x \in \mathbf{R}$.

$\therefore$ Domain of f is $\mathbf{R}$.

If $x = 0$ then $f(x) = 0$, If $x \neq 0$ then $f(x) \neq 0$.

Let $y = f(x) = \dfrac{x}{1+x^2} \Rightarrow x^2y - x + y = 0 \Rightarrow x = \dfrac{1 \pm \sqrt{1-4y^2}}{2y}$ is a real number

iff $1-4y^2 \geq 0 \Leftrightarrow (1+2y)(1-2y) \geq 0;$

$$\Rightarrow y \in \left[\frac{-1}{2}, \frac{1}{2}\right]$$

$$\therefore \text{ range of } f = \left[\frac{-1}{2}, \frac{1}{2}\right].$$

(iii) $f(x) = \sqrt{9-x^2} \in \mathbf{R} \Leftrightarrow 9-x^2 \geq 0$

$$\Leftrightarrow (3+x)(3-x) \geq 0 \Leftrightarrow x \in [-3,3].$$

$\therefore$ domain of $f = [-3,3]$.

Clearly $f(x) = \sqrt{9-x^2} \in [0, 3]$. Suppose $y \in [0, 3]$.

Then $x = \sqrt{9-y^2} \in [0, 3]$ and $f(x) = \sqrt{9-(9-y^2)} = y$.

$$\therefore \text{ range of } f = [0,3].$$

1.3.4 Some more types of functions

1. Even and odd functions : *Let* A *be a nonempty subset of* $\mathbf{R}$ *such that* $-x \in \text{A}$ *for all* $x \in \text{A}$ *and* $f : \text{A} \to \mathbf{R}$.

(i) If $f(-x) = f(x)$ for every x in A then f is called an **even function**.

(ii) If $f(-x) = -f(x)$ for every x in A then f is called an **odd function**.

Examples

(i) $f(x) = x^2$, $g(x) = \cos x$, $h(x) = |x|$ $(x \in \mathbf{R})$ are all even functions.

(ii) $f(x) = x$, $(x \in \mathbf{R})$ is an odd function.

$g(x) = \tan x$ is an odd function on $\mathbf{R} \setminus \left\{\dfrac{2n+1}{2}\pi, n \in \mathbf{Z}\right\}$.

(iii) $f(x) = x^2 + x^3$, $g(x) = \cos x + \sin x$ are neither even nor odd.

Every real valued function defined on a nonempty subset A of $\mathbf{R}$ such that $x \in \text{A} \Rightarrow -x \in \text{A}$ can be written as sum of an even and odd functions.

Consider $g(x) = \dfrac{f(x) + f(-x)}{2}$ and $h(x) = \dfrac{f(x) - f(-x)}{2}$ then g is even and h is odd since $g(x) = g(-x)$ and $h(x) = -h(-x)$. Clearly

$$f(x) = g(x) + h(x).$$

2. Polynomial function : *If n is a non negative integer, $a_0, a_1, a_2, ..., a_n$ are real numbers (at least one $a_i \neq 0$) then the function f defined on* $\mathbf{R}$ *by*

$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ *for all* $x \in \mathbf{R}$ *is called a* ***polynomial function***.

Examples

(i) $f(x) = ax + b \ (a, b \in \mathbf{R})$ is a polynomial function.

(ii) $g(x) = -7x^4 + 3x^2 + 2$ is a polynomial function.

(iii) $h(x) = k \ (0 \neq k \in \mathbf{R})$ is a polynomial function.

3. Rational function : *If f and g are polynomial functions and $g(x) \neq 0$ for all $x \in \mathbf{R}$ then the function $\frac{f}{g}$ defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ is called a* ***rational function***.

Examples: 1. $\frac{x^2 - 3x + 2}{x^2 + 1}$ is a rational function.

2. $f(x) = \frac{1}{x}, \ x \in \mathbf{R} \setminus \{0\}$ is a rational function.

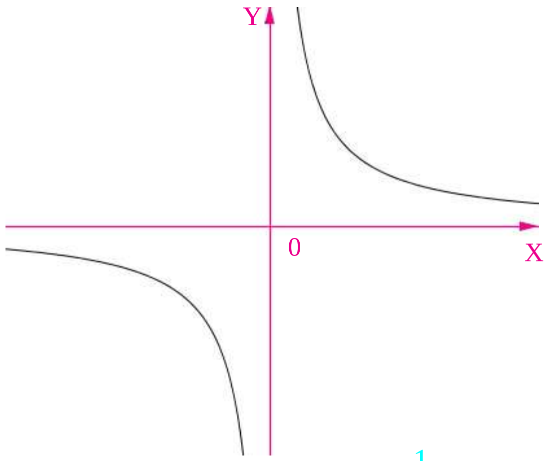


Fig. 1.4 Graph of $f(x) = \frac{1}{x}$.

4. Algebraic function: *Operations like addition, subtraction, multiplication, division and extraction of square root etc., are called algebraic operations. A function obtained by applying a finite number of algebraic operations on polynomial functions is called an* ***algebraic function***.

Examples : (i) $f(x) = \frac{3x^2 + \sqrt{9 - x^2}}{2x}, \ (x \in [-3, 3] \setminus \{0\})$.

(ii) $f(x) = \sqrt{x^2 - a^2} + 7x, \ (a > 0), \ (x \in \mathbf{R} \setminus (-a, a))$.

5. Exponential function: *The function* a^x *when* $1 \neq a > 0$ *and* x *is rational, is already defined in this chapter. This can be extended to real* x *as well, inheriting all the exponential properties. We do not present a formal definition of* a^x $(x \in \mathbf{R})$ *but assume the existence of such a (unique) function. This function is called an* ***exponential function****. Even though the definition presented in chapter 9 is slightly different, these two are equivalant. The domain of the function* a^x *is* $\mathbf{R}$ *and the range is* $\mathbf{R}^+$.

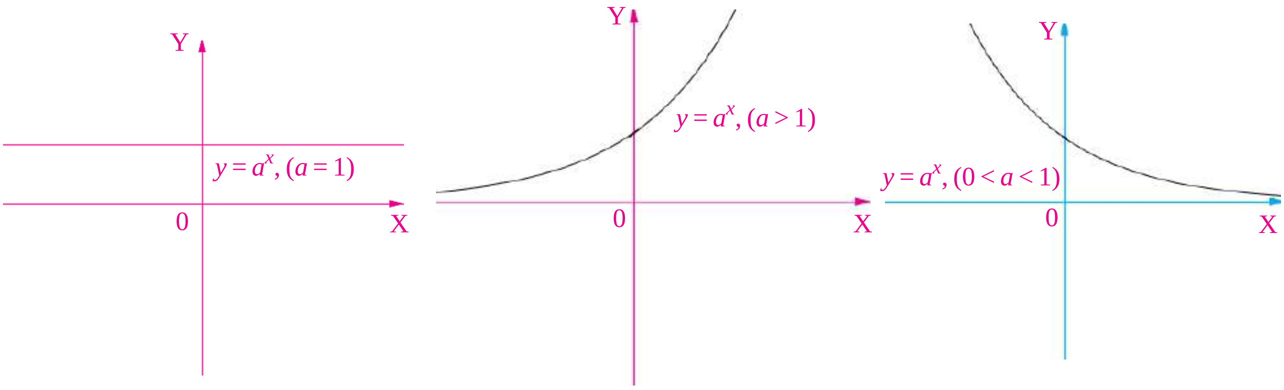


Fig. 1.5 graph of a^x

6. Logarithmic function: *If* $a > 0$, $a \neq 1$, *given* $y > 0$ *there is a unique* $x \in \mathbf{R}$ *such that* $a^x = y$. *The function defined on* $\mathbf{R}^+$ *by* $f(y) = x$, *where* $a^x = y$, *is called the* ***logarithmic function*** *to the base 'a'. This function is denoted by* $\log_a$. *Thus* $\log_a y = x$ *iff* $a^x = y$. *The logarithmic function to the base* e *is called the* ***natural logarithmic function*** *and is denoted by 'log' and also ln. Thus* $\log y = \ln y = x$ *iff* $e^x = y$. *Clearly the domain of* $\log_a$ *function is* $(0, \infty)$. *Further its range is* $\mathbf{R}$.

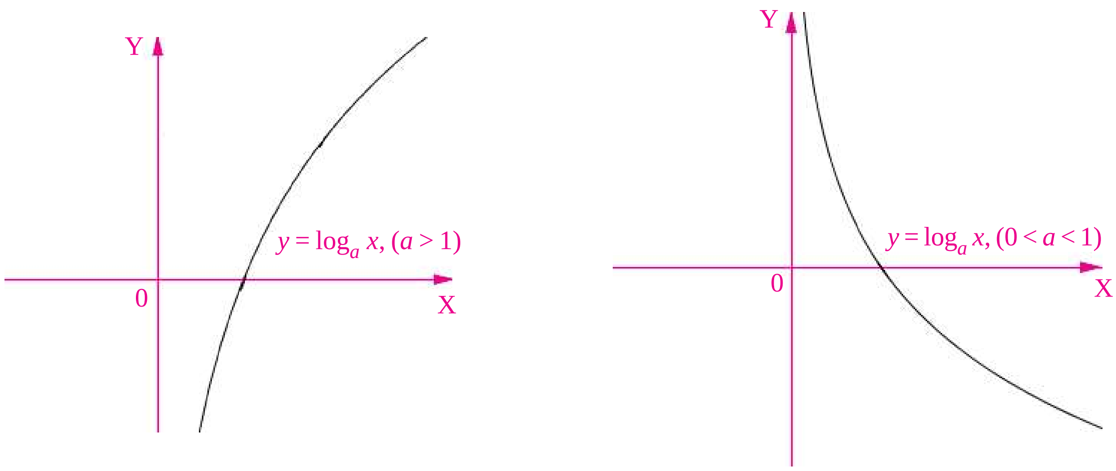


Fig. 1.6 Graph of $\log_a x$

7. Greatest Integer function: *For any real number* x, *we denote by* $[x]$, *the greatest integer less than or equal to* x. *For example* $[1.72] = 1, [-3.41] = -4$, $[0.22] = 0$, $[-0.71] = -1$.

The function $f:\mathbf{R}\to\mathbf{R}$ *defined by* $f(x)=[x]$ *for all* $x\in\mathbf{R}$ *is called the* ***greatest integer function****. The domain of the greatest integer function is* $\mathbf{R}$ *and the range is the set* $\mathbf{Z}$ *of all integers.*

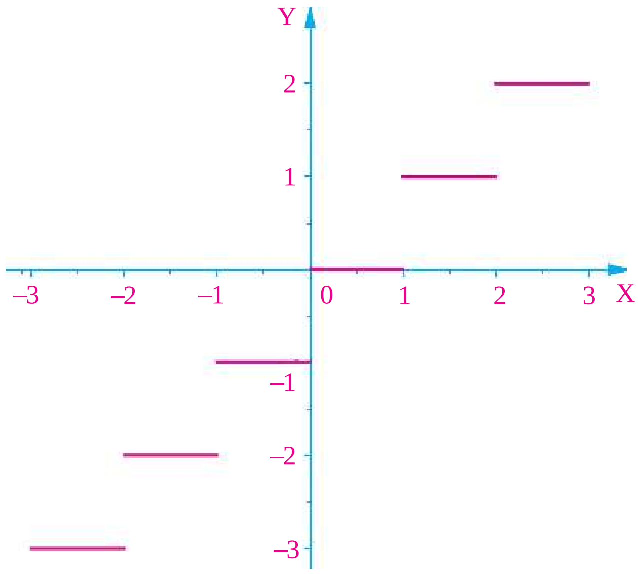


Fig. 1.7 Graph of greatest integer function

8. Modulus function : *The function* $f:\mathbf{R}\to\mathbf{R}$ *defined by* $f(x) = |x|$ *for each* $x\in\mathbf{R}$ *is called* ***modulus function****. For each non-negative value of x,* $f(x)$ *is equal to x. But for negative values of x, the value of* $f(x)$ *is the negative of the value of x i.e.,*

$$f(x)=\begin{cases} x, & x\geq 0 \\ -x, & x<0 \end{cases}$$

The graph is

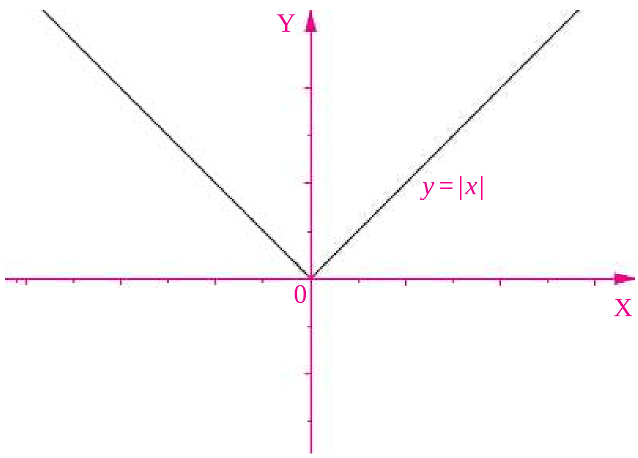


Fig. 1.8 $f(x) = |x|$

9. Signum function: *The function* $f : \mathbf{R} \to \mathbf{R}$ *defined by*

$$sgn(x) = f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$ *is called* ***signum function***. *The domain is* **R** *and range is* $\{-1, 0, 1\}$.

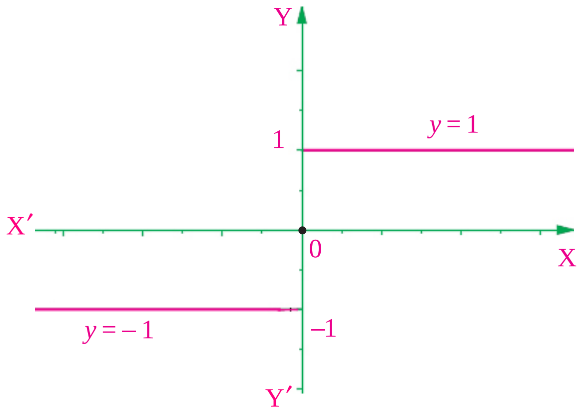


Fig. 1.9 Graph of Signum function

1.3.5 Solved Problems

1. Problem: *If* $f(x) = x^2$ *and* $g(x) = |x|$, *find the following functions.*

(i) $f + g$, (ii) $f - g$, (iii) fg, (iv) $2f$, (v) f^2, (vi) $f + 3$

Solution: Given that $f(x) = x^2$, $g(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$, domain f = domain g = **R**. Hence the domain of all the functions (i) through (vi) is **R.**

(i) $(f + g)(x) = f(x) + g(x) = x^2 + |x| = \begin{cases} x^2 + x, & x \geq 0 \\ x^2 - x, & x < 0 \end{cases}$

(ii) $(f - g)(x) = f(x) - g(x) = x^2 - |x| = \begin{cases} x^2 - x, & x \geq 0 \\ x^2 + x, & x < 0 \end{cases}$

(iii) $(fg)(x) = f(x)g(x) = x^2|x| = \begin{cases} x^3, & x \geq 0 \\ -x^3, & x < 0 \end{cases}$

(iv) $(2f)(x) = 2f(x) = 2x^2$.

(v) $f^2(x) = (f(x))^2 = (x^2)^2 = x^4$.

(vi) $(f + 3)(x) = f(x) + 3 = x^2 + 3$.

2. Problem: *Determine whether the following functions are even or odd.*

(i) $f(x) = a^x - a^{-x} + \sin x$, (ii) $f(x) = x\left(\frac{e^x - 1}{e^x + 1}\right)$,

(iii) $f(x) = \log(x + \sqrt{x^2+1})$

Solution: Clearly in all the cases domain $f = \mathbf{R}$

(i) We have $f(x) = a^x - a^{-x} + \sin x$

$$\therefore f(-x) = a^{-x} - a^x + \sin(-x) = a^{-x} - a^x - \sin x = -(a^x - a^{-x} + \sin x) = -f(x).$$

$\therefore f$ is an odd function.

(ii) $f(x) = x\left(\dfrac{e^x - 1}{e^x + 1}\right)$

$$f(-x) = (-x)\left(\frac{e^{-x}-1}{e^{-x}+1}\right) = -x\left(\frac{1-e^x}{1+e^x}\right) = x\left(\frac{e^x-1}{e^x+1}\right) = f(x).$$

$\therefore f$ is an even function.

(iii) $f(x) = \log(x + \sqrt{x^2+1}) \Rightarrow f(-x) = \log(-x + \sqrt{x^2+1})$

$$\therefore f(-x) = \log\left[\frac{(x+\sqrt{x^2+1})\,(-x+\sqrt{x^2+1})}{(x+\sqrt{x^2+1})}\right]$$

$$= \log\left(\frac{x^2+1-x^2}{x+\sqrt{x^2+1}}\right) = \log(x+\sqrt{x^2+1})^{-1}$$

$$\therefore f(-x) = -\log(x+\sqrt{x^2+1}) = -f(x).$$

$\therefore f$ is an odd function.

3. Problem: *Find the domains of the following real valued functions.*

(i) $f(x) = \dfrac{1}{\sqrt{[x]^2 - [x] - 2}}$

(ii) $f(x) = \log(x - [x])$

(iii) $f(x) = \sqrt{\log_{10}\left(\dfrac{3-x}{x}\right)}$

(iv) $f(x) = \sqrt{x+2} + \dfrac{1}{\log_{10}(1-x)}$

(v) $f(x) = \dfrac{\sqrt{3+x} + \sqrt{3-x}}{x}$

Solution

(i) $f(x) = \dfrac{1}{\sqrt{[x]^2 - [x] - 2}} \in \mathbf{R} \Leftrightarrow [x]^2 - [x] - 2 > 0$

$$\Leftrightarrow ([x]+1)([x]-2) > 0$$

$$\Leftrightarrow [x] < -1 \text{ (or) } [x] > 2.$$

But $[x] < -1 \Rightarrow [x] = -2, -3, -4 \ldots\ldots\ldots\ldots \Rightarrow x < -1$

$[x] > 2 \Rightarrow [x] = 3, 4, 5, \ldots\ldots\ldots\ldots \Rightarrow x \geq 3$

$\therefore$ Domain of $f = (-\infty, -1) \cup [3, \infty) = \mathbf{R} \setminus [-1, 3)$.

(ii) $f(x) = \log(x - [x]) \in \mathbf{R} \Leftrightarrow x - [x] > 0 \Leftrightarrow x > [x]$

$\Leftrightarrow x$ is a non-integer

$\therefore$ Domain of f is $\mathbf{R} \setminus \mathbf{Z}$.

(iii) $f(x) = \sqrt{\log_{10}\left(\frac{3-x}{x}\right)} \in \mathbf{R} \Leftrightarrow \log_{10}\left(\frac{3-x}{x}\right) \geq 0$ and $\frac{3-x}{x} > 0$

$\Leftrightarrow \frac{3-x}{x} \geq 10^0 = 1$ and $3 - x > 0,\ x > 0$

$\Leftrightarrow 3 - x \geq x$ and $0 < x < 3$

$\Leftrightarrow x \leq 3/2$ and $0 < x < 3$

$\Leftrightarrow x \in \left(-\infty, \frac{3}{2}\right] \cap (0,\ 3) = \left(0, \frac{3}{2}\right]$

$\therefore$ Domain of f is $\left(0, \frac{3}{2}\right]$.

(iv) $f(x) = \sqrt{x+2} + \frac{1}{\log_{10}(1-x)} \in \mathbf{R} \Leftrightarrow x + 2 \geq 0$ and $1 - x > 0$ and $1 - x \neq 1$

$\Leftrightarrow x \geq -2$ and $1 > x$ and $x \neq 0$

$\Leftrightarrow x \in [-2, \infty) \cap (-\infty, 1) \setminus \{0\} \Leftrightarrow x \in [-2, 1) \setminus \{0\}$

$\therefore$ Domain of f is $[-2, 1) \setminus \{0\}$.

(v) $f(x) = \frac{\sqrt{3+x} + \sqrt{3-x}}{x} \in \mathbf{R} \Leftrightarrow 3 + x \geq 0,\ 3 - x \geq 0,\ x \neq 0$

$\Leftrightarrow -3 \leq x \leq 3,\ x \neq 0$

$\Leftrightarrow x \in [-3, 3] \setminus \{0\}$

$\therefore$ Domain of f is $[-3, 3] \setminus \{0\}$.

Exercise 1(c)

I. 1. Find the domains of the following real valued functions.

(i) $f(x) = \dfrac{1}{(x^2-1)(x+3)}$ (ii) $f(x) = \dfrac{2x^2-5x+7}{(x-1)(x-2)(x-3)}$

(iii) $f(x) = \dfrac{1}{\log(2-x)}$ (iv) $f(x) = |x-3|$

(v) $f(x) = \sqrt{4x-x^2}$ (vi) $f(x) = \dfrac{1}{\sqrt{1-x^2}}$

(vii) $f(x) = \dfrac{3^x}{x+1}$ (viii) $f(x) = \sqrt{x^2-25}$

(ix) $f(x) = \sqrt{x-[x]}$ (x) $f(x) = \sqrt{[x]-x}$

2. Find the ranges of the following real valued functions.

(i) $\log|4-x^2|$ (ii) $\sqrt{[x]-x}$ (iii) $\dfrac{\sin\pi[x]}{1+[x]^2}$

(iv) $\dfrac{x^2-4}{x-2}$ (v) $\sqrt{9+x^2}$

3. If f and g are real valued functions defined by $f(x)=2x-1$ and $g(x)=x^2$ then find

(i) $(3f-2g)(x)$ (ii) $(fg)(x)$ (iii) $\left(\dfrac{\sqrt{f}}{g}\right)(x)$

(iv) $(f+g+2)(x)$

4. If $f=\{(1,2),(2,-3),(3,-1)\}$ then find

(i) $2f$ (ii) $2+f$ (iii) f^2 (iv) $\sqrt{f}$

II. 1. Find the domains of the following real valued functions.

(i) $f(x)=\sqrt{x^2-3x+2}$ (ii) $f(x)=\log(x^2-4x+3)$

(iii) $f(x)=\dfrac{\sqrt{2+x}+\sqrt{2-x}}{x}$ (iv) $f(x)=\dfrac{1}{\sqrt[3]{(x-2)}\log_{(4-x)}10}$

(v) $f(x)=\sqrt{\dfrac{4-x^2}{[x]+2}}$ (vi) $f(x)=\sqrt{\log_{0.3}(x-x^2)}$

(vii) $f(x)=\dfrac{1}{x+|x|}$

2. Prove that the real valued function $f(x) = \frac{x}{e^x - 1} + \frac{x}{2} + 1$ is an even function on $\mathbf{R} \setminus \{0\}$.

3. Find the domain and range of the following functions.

(i) $f(x) = \frac{\tan \pi[x]}{1 + \sin \pi[x] + [x^2]}$ (ii) $f(x) = \frac{x}{2 - 3x}$ (iii) $f(x) = |x| + |1 + x|$

Key Concepts

- If $f : A \to B$ is a function then $f(A) = \{f(a) \mid a \in A\}$ is called the range f. It is a subset of B, and is denoted by Range f.
- $f : A \to B$ is an injection $\Leftrightarrow$ $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ imply $a_1 = a_2$.
- $f : A \to B$ is a surjection $\Leftrightarrow$ range f = codomain B $\Leftrightarrow$ for any $b \in B$ there exists atleast one $a \in A$ such that $f(a) = b$.
- $f : A \to B$ is a bijection $\Leftrightarrow$ f is both an injection and a surjection.
- If $f : A \to B$ is a bijection then the relation $f^{-1} = \{(b, a) \mid (a, b) \in f\}$ is a bijection from B to A and is called the inverse function of f.
- Let $f : A \to B$, $g : B \to C$ be functions then $(gof) : A \to C$ is a function and $(gof)(a) = g(f(a))$ for all $a \in A$.
- If $f : A \to B$, $g : B \to C$ are bijections so is $(gof) : A \to C$ and $(gof)^{-1} = f^{-1}og^{-1}$.
- If $f : A \to B$ is a bijection, then $fof^{-1} = I_B$ and $f^{-1}of = I_A$.
- If $f : A \to B$, $g : B \to C$ are such that $gof = I_A$, $fog = I_B$ then f is a bijection and $g = f^{-1}$.
- Let A be a nonempty subset of $\mathbf{R}$ such that $-x \in A$ for all $x \in A$ and $f : A \to \mathbf{R}$.

  (i) If $f(-x) = f(x)$ for all $x \in A$ then f is called an even function.

  (ii) If $f(-x) = -f(x)$ for all $x \in A$ then f is called an odd function.

Historical Note

The history of the term **"Function"** furnishes an interesting example of the enthusiasm in mathematicians to modify, refine and generalize their concepts.

The word **"Function"** seems to have been known to ***Descartes*** (1596 - 1650) in 1637, who employed the term simply to mean some positive integral powers, x^n, of a variable x. Somewhat later, ***Leibnitz*** (1646 - 1716) employed the term to denote any quantity connected with a curve, such as the coordinates of a point on the curve, the slope of the curve etc. ***Johann Bernoulli*** (1667 - 1748) regarded a function as any expression made up of a variable and some constants and ***Euler*** (1707 - 1783) gave a symbolic representation as $f(x)$ to a function. Euler's concept remains unchanged till ***Fourier*** (1768 - 1830) has modified the earlier definition of a function in his investigations of trigonometric series. These series involve a more general type of relationship between variables that had previously been studied and have become instrumental in his attempt to furnish the present definition of function broad enough to encompass such relationships by ***Lejeune Dirichlet*** (1805 - 1859).

Answers

Exercise 1(a)

I. **1.** (i) 5 (ii) 2 (iii) –2.5 (iv) 1 (v) Not defined

5. {3, 1, 7}

6. $\left\{\frac{1}{2}, 1, \frac{7}{4}, \frac{13}{5}\right\}$

II. **1.** (i) f is not a surjection (ii) g is a surjection

2. (i) bijection (ii) bijection

(iii) bijection (iv) bijection

(v) not an injection but a surjection (vi) neither injection nor surjection

3. $a = 2$; $b = -1$ **5.** 2 **6.** $a = \pm 1$, $b = 1$

Exercise 1(b)

I. **1.** $f^{-1}(x)=\log_e x,\ g^{-1}(x)=e^x$

3. **(i)** $18x^2-24x+11$ **(ii)** $6x^2+7$ **(iii)** 21

(iv) 2653

4. **(i)** $9x^2+5$ **(ii)** 14 **(iii)** $36a^2-120a+101$

5. $\frac{1}{\sqrt{x}}$ **6.** x **7.** 2

8. **(i)** $\frac{x-b}{a}$ **(ii)** $\log_5 x$ **(iii)** 2^x

10. $\frac{\sqrt{1+4\log_2 x}}{2}$

II. **3. (i)** $\frac{25}{9}$ **(ii)** $9x^2-30x+26$ **5.** $\left(\frac{x-7}{2}\right)^{1/3}$

6. $x=0,2$ **7.** $f(x), x$

Exercise 1(c)

I. **1.** **(i)** $\mathbf{R}\setminus\{-1, 1, -3\}$ **(ii)** $\mathbf{R}\setminus\{1, 2, 3\}$

(iii) $(-\infty,2)\setminus\{1\}$ **(iv)** $\mathbf{R}$

(v) $[0,4]$ **(vi)** $(-1, 1)$

(vii) $\mathbf{R}\setminus\{-1\}$ **(viii)** $\mathbf{R}\setminus(-5, 5,)$

(ix) $\mathbf{R}$ **(x)** $\mathbf{Z}$

2. **(i)** $\mathbf{R}$ **(ii)** $\{0\}$

(iii) $\{0\}$ **(iv)** $\mathbf{R}\setminus\{4\}$

(v) $[3, \infty)$

3. **(i)** $-2x^2+6x-3$ **(ii)** $2x^3-x^2$

(iii) $\frac{\sqrt{2x-1}}{x^2}$ **(iv)** $(x+1)^2$

4. (i) $\{(1,4),\ (2,-6),(3,-2)\}$ (ii) $\{(1,4),\ (2,-1),(3,1)\}$

(iii) $\{(1,4),\ (2,9),(3,1)\}$ (iv) $\{(1,\sqrt{2})\}$

II. 1. (i) $\mathbf{R} \setminus (1,2)$ (ii) $\mathbf{R} \setminus [1, 3]$

(iii) $[-2, 2] \setminus \{0\}$ (iv) $(-\infty, 4) \setminus \{2, 3\}$

(v) $(-\infty,-2)\cup[-1,2]$, (vi) $(0,1)$

(vii) $(0, \infty)$

3. (i) Domain $\mathbf{R}$, range $\{0\}$

(ii) Domain $\mathbf{R} \setminus \left\{\frac{2}{3}\right\}$, Range $\mathbf{R} \setminus \left\{\frac{-1}{3}\right\}$

(iii) Domain $\mathbf{R}$, range $[1,\infty)$

Chapter 2

Mathematical Induction

"Analysis and natural philosophy owe their most important discoveries to this fruitful means, which is called induction"

– Laplace

Introduction

A famous Italian Mathematician, Peano defined a function $f : \mathbf{N} \to \mathbf{N}$ as $f(n) = n + 1$ which is known as Peano successor function. He obtained some algebraic properties of the set **N** of all natural numbers by using this function f in his axiomatic approach. One of his axioms is known as Inductive axiom or Induction Theorem.

To understand the basic principles of mathematical induction consider the following simple example.

Suppose a set of bicycles are placed, very closely adjacent to each other.

When the first bicycle is pushed in a particular direction, all the bicycles will fall in that direction.

Laplace

(1749 - 1827)

Pierre Simon de Laplace *was a French mathematician and astronomer whose work was pivotal to the development of mathematical astronomy. His most outstanding work was done in the fields of celestial mechanics, probability, differential equations, and geodesy. His five volume work on celestial mechanics earned him the title of the* ***Newton of France****.*

To be absolutely sure that all the bicycles will fall, it is sufficient to know that

(a) the first bicycle falls and

(b) in the event that any bicycle falls, its successor necessarily falls.

This is the underlying principle of mathematical induction.

Mathematical Induction is a powerful tool frequently used to establish the validity of statements that are given in terms of the natural numbers.

The inductive aspect is concerned with the search for facts by observation and experimentation. For example, we all know the fact that "Sun rises in the east". How can we say this happens always ? By observing this phenomenon from ages, we conclude that this goes on. Thus, we arrive at a conjecture for a general rule by inductive reasoning.

2.1 Principles of Mathematical Induction & Theorems

Here under we state the well-ordering principle of the positive integer, which can be used for the proof of principle of finite mathematical induction. However, we do not attempt to prove these theorems at this stage. Students who aspire to choose mathematics as major subject at the degree level have an opportunity to learn the proof of both these theorems, the well-ordering principle and the principle of finite mathematical induction.

2.1.1 Well - Ordering principle

Any non-empty set of positive integers has a least element.

2.1.2 Principle of finite mathematical induction

Let S be a subset of **N** such that

1. $1 \in S$
2. For any $k \in \mathbf{N}$, $k \in S \Rightarrow k+1 \in S$.

Then $S = \mathbf{N}$.

2.1.3 Equivalent forms of principle of finite mathematical induction

Principle of finite mathematical induction has a good number of equivalent forms which are used in appropriate occasions. Three of them are stated here in 2.1.4, 2.1.5 and 2.1.6. We present proof for 2.1.5 as it is an immediate consequence of 2.1.2, the principle of finite mathematical induction and leave the others as exercises.

2.1.4 Statement : For each $n \in \mathbf{N}$, let P(n) be a statement. Suppose that

(i) P(1) is true.

(ii) for any $k \in \mathbf{N}$, if P(k) is true, then P($k+1$) is true.

Then P(n) is true for all $n \in \mathbf{N}$.

2.1.5 Principle of complete mathematical induction

Let S be a subset of **N** such that

(i) $1 \in S$

(ii) for any $k \in \mathbf{N}$, $\{1, 2, 3, ..., k\} \subseteq S \Rightarrow k+1 \in S$

Then $S = \mathbf{N}$.

Proof: Let $T = \{m \in \mathbf{N} : 1, 2, ..., m \in S\}$

Then $1 \in S \Rightarrow 1 \in T$ and

$$n \in T \Rightarrow 1, 2, ..., n \in S$$
$$\Rightarrow (n+1) \in S$$
$$\Rightarrow (n+1) \in T$$

By the principle of finite induction (2.1.2), it follows that $T = \mathbf{N}$.

$\therefore$ $N = T \subseteq S$. But by hypothesis $S \subseteq \mathbf{N}$.

Accordingly $S = \mathbf{N}$.

2.1.6 Statement : For each $n \in \mathbf{N}$, let P(n) be a statement. Suppose that

(i) P(1) is true

(ii) for any $k \in \mathbf{N}$, if P(1), P(2), ... , P(k) are true, then P($k+1$) is true.

Then P(n) is true for all $n \in \mathbf{N}$.

It may happen that statements P(n) are false for certain natural numbers but they are true for all $n \geq n_0$ for some particular n_0.

For example, the statement $P(n) = (n-1)(n-3)(n-5)$ is a positive integer is true for all $n \geq 6$ but not true when $n = 1$ or $n = 3$ or $n = 4$ or $n = 5$. However it is true for $n = 2$ also. The principle of mathematical induction can be modified to deal with this situation. We will formulate the modified principle, without proof.

2.1.7 Principle of mathematical induction (Modified version)

Let $n_0 \in \mathbf{N}$ and let $P(n)$ be a statement for each natural number $n \geq n_0$. Suppose that

(i) The statement $P(n_0)$ is true.

(ii) For all $k \geq n_0$, $P(k)$ is true $\Rightarrow$ $P(k+1)$ is true.

Then $P(n)$ is true for all $n \geq n_0$.

2.1.8 Steps to prove a statement using the principle of mathematical induction

The starting point or basis of induction is usually 1, but could be negative integer, positive integer or zero. Normally we expect to prove that $P(k) \Rightarrow P(k+1)$. So there are 3 steps to prove a statement using the principle of mathematical induction.

1. **Basis of induction** : Show that $P(1)$ is true.
2. **Inductive hypothesis** : For $k \geq 1$, assume that $P(k)$ is true.
3. **Inductive step** : Show that $P(k+1)$ is true on the basis of the inductive hypothesis.

Let us consider an example from which we observe that the principle of mathematical induction is only a method of proof for a known or guessed or predicted formula and it is not a tool for finding such formula.

2.1.9 Example

Let $S(n) = 1 + 2 + 3 + \cdots + n$.

Let us examine a few values for $S(n)$ and list them in the following table:

n	1	2	3	4	5	6	7	8	9	10	11
$S(n)$	1	3	6	10	15	21	28	36	45	55	66

To guess a formula for $S(n)$ may not be an easy task. But we can observe the following pattern :

$$2\,S(1) = 2 = 1.2$$

$$2\,S(2) = 6 = 2.3$$

$$2\,S(3) = 12 = 3.4$$

$$2\,S(4) = 20 = 4.5, \text{ and so on.}$$

This leads us to conjecture that

$$2\,S(n) = n(n+1) \text{ so that } S(n) = \frac{n(n+1)}{2}$$

i.e., $1+2+3+\cdots+n=\frac{n(n+1)}{2}$.

Now let us use mathematical induction to prove the above formula.

Let P(n) be the statement : The sum S(n) of the first n positive integers is equal to $\frac{n(n+1)}{2}$.

1. Basis of induction : Since $S(1) = 1 = \frac{1(1+1)}{2}$, the formula is true for $n=1$.

2. Inductive hypothesis : Assume the statement P(n) is true for $n=k$.

i.e., $S(k)=1+2+3+\cdots+k=\frac{k(k+1)}{2}$.

3. Inductive step : To show that the formula is true for $n=k+1$.

i.e., to show that $S(k+1)=\frac{(k+1)(k+2)}{2}$.

We observe that $S(k+1) = 1+2+3+\cdots+k+(k+1)$
$= S(k)+(k+1)$

Since $S(k)=\frac{k(k+1)}{2}$, by the inductive hypothesis,

we have $$S(k+1) = \frac{k(k+1)}{2}+(k+1) = \frac{(k+1)(k+2)}{2}$$

Therefore the formula holds for $n=(k+1)$.

$\therefore$ By the principle of mathematical induction, P(n) is true for all $n\in \mathbf{N}$.

i.e., the formula, $1+2+3+\cdots+n=\frac{n(n+1)}{2}$ is true for all $n\in\mathbf{N}$.

2.2 Applications of Mathematical Induction

Mathematical induction is very useful in proving many theorems and statements. For example, it is useful in proving Binomial theorem, Leibnitz theorem for finding n^{th} order derivative of the product of two functions and evaluation of some integrals etc.

We now illustrate the utility of mathematical induction in proving some statements.

2.2.1 Solved Problems

1. Problem: *Use mathematical induction to prove the statement,*

$$1^3+2^3+3^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}, \forall\, n\in \mathbf{N}.$$

Solution: Let P(n) be the statement :

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2\,(n+1)^2}{4}$$

Since $1 = \frac{1\,(1+1)^2}{4}$ the formula is true for $n = 1$.

Assume that statement P(n) is true for $n = k$, $k \geq 1$

$$\text{i.e., } 1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2\,(k+1)^2}{4}.$$

We show that the formula is true for $n = k + 1$,

i.e., we show that $S(k+1) = \frac{(k+1)^2\,(k+2)^2}{4}$ (where $S(k) = 1^3 + 2^3 + \cdots + k^3$)

We observe that $S(k+1) = 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3$

$$= S(k) + (k+1)^3.$$

Since, $S(k) = \frac{k^2\,(k+1)^2}{4}$,

we have
$$\begin{aligned} S\,(k+1) &= S(k) + (k+1)^3 \\ &= \frac{k^2\,(k+1)^2}{4} + (k+1)^3 \\ &= \frac{(k+1)^2}{4}\left[k^2 + 4\,(k+1)\right] \\ &= \frac{(k+1)^2\,(k+2)^2}{4} \end{aligned}$$

∴ The formula holds for $n = k + 1$.

∴ By the principle of mathematical induction, P(n) is true for all $n \in \mathbf{N}$.

i.e., the formula $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2\,(n+1)^2}{4}$ is true for all $n \in \mathbf{N}$.

2. Problem: *Use mathematical induction to prove the statement,*

$$\sum_{k=1}^{n} (2k-1)^2 = \frac{n\,(2n-1)\,(2n+1)}{3} \quad \text{for all } n \in \mathbf{N}.$$

Solution: Let P(n) be the statement :

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n\,(2n-1)\,(2n+1)}{3}.$$

Let S(n) be the sum $1^2 + 3^2 + \cdots + (2n-1)^2$.

Since $S(1) = 1 = \frac{1\,(2-1)\;(2+1)}{3}$, the formula is true for $n = 1$.

Assume that the statement P(n) is true for $n = k$, $k \geq 1$.

i.e., $S(k) = 1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 = \frac{k\,(2k-1)\,(2k+1)}{3}$.

We show that the formula is true for $n = k + 1$,

i.e., we show that $S(k+1) = \frac{(k+1)\,(2k+1)\,(2k+3)}{3}$.

We observe that $S(k+1) = 1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 + (2k+1)^2$

$$= S(k) + (2k+1)^2$$

Since $S(k) = \frac{k\,(2k-1)\,(2k+1)}{3}$,

we have
$$S(k+1) = S(k) + (2k+1)^2$$
$$= \frac{k\,(2k-1)\,(2k+1)}{3} + (2k+1)^2$$
$$= (2k+1)\left[\frac{k\,(2k-1)}{3} + (2k+1)\right]$$
$$= (2k+1)\left[\frac{2k^2 + 5k + 3}{3}\right]$$
$$= \frac{(2k+1)\,(k+1)\,(2k+3)}{3}$$

$\therefore$ $S(k+1) = \frac{(k+1)\,(2k+1)\,(2k+3)}{3}$.

$\therefore$ The formula holds for $n = k + 1$.

$\therefore$ By the principle of mathematical induction, P(n) is true for all $n \in \mathbf{N}$.

i.e., the formula $\sum_{k=1}^{n} (2k-1)^2 = \frac{n\,(2n-1)\,(2n+1)}{3}$ is true for all $n \in \mathbf{N}$.

3. Problem: *Use mathematical induction to prove the statement,*

$$2 + 3.2 + 4.2^2 + \cdots \text{ upto } n \text{ terms} = n.\,2^n, \;\forall\, n \in \mathbf{N}.$$

Solution: Let P(n) be the statement :

$$2 + 3.2 + 4.2^2 + \cdots + (n+1)\,2^{n-1} = n.2^n$$

Let S(n) be the sum $2 + 3.2 + 4.2^2 + \ldots + (n+1)\,2^{n-1}$.

Since $S(1) = 2 = 1.\,2^1$, the formula is true for $n = 1$.

Assume that the statement P(n) is true for $n = k, k \geq 1$.

i.e., $S(k) = 2 + 3.2 + 4.2^2 + \cdots + (k+1)\, 2^{k-1} = k.2^k$

We show that the formula is true for $n = k + 1$

i.e., we show that $S(k+1) = (k+1).2^{k+1}$.

We observe that $S(k+1) = 2 + 3.2 + 4.2^2 + \cdots + (k+1)\, 2^{k-1} + (k+2)\, 2^k$

$$= S(k) + (k+2)\, 2^k.$$

Since , $S(k) = k\, 2^k$, we have, $S(k+1) = S(k) + (k+2)\, 2^k$

$$= k.2^k + (k+2)\, 2^k$$

$$= 2^k (k + k + 2)$$

$$= (k+1)\, 2^{k+1}.$$

$\therefore$ The formula holds for $n = k + 1$.

$\therefore$ By the principle of mathematical induction, P(n) is true for all $n \in \mathbf{N}$.

i.e., the formula $2 + 3.2 + 4.2^2 + \cdots + (n+1)\, 2^{n-1} = n\, 2^n$ is true for all $n \in \mathbf{N}$.

4. Problem: *Show that,* $\forall\ n \in \mathbf{N}$, $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots$ *upto n terms* $= \frac{n}{3n+1}$.

Solution: 1, 4, 7, ... are in Arithmetic Progression whose n^{th} term is $3n - 2$.

4, 7, 10, ... are also in Arithmetic Progression whose n^{th} term is $3n + 1$.

$\therefore$ The n^{th} term in the given series is $\frac{1}{(3n-2)(3n+1)}$.

Let P(n) be the statement :

$$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{(3n+1)}$$

and let S(n) be the sum on the left hand side.

Since $S(1) = \frac{1}{1.4} = \frac{1}{3.1+1}$, the formula is true for $n = 1$.

Assume that the statement P(n) is true for $n = k, k \geq 1$.

i.e., $S(k) = \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1}$.

We show that the formula is true for $n = k + 1$,

i.e., we show that $S(k+1) = \frac{k+1}{3k+4}$.

We observe that

$$S(k+1) = \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots + \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3k+1)(3k+4)}$$

$$= S(k) + \frac{1}{(3k+1)(3k+4)}.$$

Since $S(k) = \frac{k}{3k+1}$ we have, $S(k+1) = S(k) + \frac{1}{(3k+1)(3k+4)}$

$$= \frac{k}{(3k+1)} + \frac{1}{(3k+1)(3k+4)}$$

$$= \frac{k(3k+4)+1}{(3k+1)(3k+4)}$$

$$= \frac{(k+1)(3k+1)}{(3k+1)(3k+4)}$$

$$= \frac{k+1}{3k+4}.$$

$\therefore$ The formula holds for $n = k + 1$.

$\therefore$ By the principle of mathematical induction, P(n) is true for all $n \in \mathbf{N}$.

5. Problem: *Use mathematical induction to prove that $2n - 3 \leq 2^{n-2}$ for all $n \geq 5$, $n \in$* **N**.

Solution: Let P(n) be the statement : $2n - 3 \leq 2^{n-2}$, $\forall\ n \geq 5$, $n \in \mathbf{N}$.

Here we note that the basis of induction is 5.

Since $2.5 - 3 \leq 2^{5-2}$, the statement is true for $n = 5$.

Assume the statement is true for $n = k$, $k \geq 5$.

i.e., $2k - 3 \leq 2^{k-2}$, for $k \geq 5$.

We show that the statement is true for $n = k + 1$, $k \geq 5$

i.e., $[2(k+1) - 3] \leq 2^{(k+1)-2}$, for $k \geq 5$.

We observe that $[2(k+1) - 3] = (2k - 3) + 2$

$\leq 2^{k-2} + 2$, (By inductive hypothesis)

$\leq 2^{k-2} + 2^{k-2}$ for $k \geq 5$

$= 2.2^{k-2}$

$= 2^{(k+1)-2}$

$\therefore$ The statement P(n) is true for $n = k + 1$, $k \geq 5$.

$\therefore$ By the principle of mathematical induction, the statement is true for all $n \geq 5$, $n \in \mathbf{N}$.

6. Problem: *Use mathematical induction to prove that* $(1 + x)^n > 1 + nx$ *for* $n \geq 2, x > -1, x \neq 0$.

Solution: Let the statement P(n) be : $(1 + x)^n > 1 + nx$.

Here we note that the basis of induction is 2 and that $x \neq 0,\ x > -1$

$$\Rightarrow\ 1 + x > 0.$$

Since $(1 + x)^2 = 1 + 2x + x^2 > 1 + 2x$, the statement is true for $n = 2$.

Assume that the statement is true for $n = k$, $k \geq 2$.

i.e., $(1 + x)^k > 1 + kx$ for $k \geq 2$

We show that the statement is true for $n = k + 1$,

i.e., $(1 + x)^{k+1} > 1 + (k + 1)x$.

We observe that
$$\begin{aligned}(1 + x)^{k+1} &= (1 + x)^k . (1 + x)\\ &> (1 + kx) . (1 + x), \quad \text{(By inductive hypothesis)}\\ &= 1 + (k + 1)x + kx^2\\ &> 1 + (k + 1)x, \quad\quad \text{(since } kx^2 > 0)\end{aligned}$$

$\therefore$ The statement is true for $n = k + 1$.

$\therefore$ By the principle of mathematical induction, the statement P(n) is true for all $n \geq 2$.

i.e., $(1 + x)^n > 1 + nx,\ \forall\, n \geq 2,\ x > -1,\ x \neq 0$.

2.3 Problems on divisibility

In the following problems, we illustrate the method of using mathematical induction to prove the statements on divisibility.

2.3.1 Solved Problems

1. Problem: *If* x *and* y *are natural numbers and* $x \neq y$, *using mathematical induction, show that* $x^n - y^n$ *is divisible by* $x - y$, *for all* $n \in \mathbf{N}$.

Solution: Let P(n) be the statement :

$x^n - y^n$ is divisible by $x - y$.

Since $x^1 - y^1 = x - y$ is divisible by $x - y$, the statement is true for $n = 1$.

Assume that the statement P(n) is true for $n = k, k \geq 1$.

i.e., $x^k - y^k$ is divisible by $x - y$.

Then $x^k - y^k = (x - y)\,p$, where p is the quotient when $x^k - y^k$ is divided by $x - y$. ... (1)

We show that the statement is true for $n = k + 1$,

i.e., we show that $x^{k+1} - y^{k+1}$ is divisible by $x - y$.

From (1), we have $x^k - y^k = (x - y)\, p$

$$\therefore \quad x^k = (x - y)\, p + y^k .$$

$$\therefore \quad x^{k+1} = (x - y)\, p\, x + y^k \cdot x$$

$$\therefore \quad x^{k+1} - y^{k+1} = (x - y)\, p\, x + y^k x - y^{k+1}$$

$$= (x - y)\, p\, x + y^k (x - y).$$

$$= (x - y)\, (p\, x + y^k)$$

$\therefore$ $x^{k+1} - y^{k+1}$ is divisible by $x - y$.

$\therefore$ The statement P(n) is true for $n = k + 1$.

$\therefore$ By the principle of mathematical induction, P(n) is true for all $n \in \mathbf{N}$

i.e., $x^n - y^n$ is divisible by $x - y$ for all $n \in \mathbf{N}$.

2. Problem: *Using mathematical induction, show that* $x^m + y^m$ *is divisible by* $x + y$, *if* m *is an odd natural number and* x, y *are natural numbers.*

Solution: Since m is an odd natural number, there exists a non negative integer n such that $m = 2n + 1$.

Let P(n) be the statement : $x^{2n+1} + y^{2n+1}$ is divisible by $x + y$.

Since $x^1 + y^1 = x + y$ is divisible by $x + y$, the statement is true for $n = 0$ and $x^{2.1+1} + y^{2.1+1} = x^3 + y^3 = (x + y)\,(x^2 - x\,y + y^2)$ is divisible by $x + y$, the statement is true for $n = 1$.

Assume that the statement P(n) is true for $n = k$, $k \geq 1$.

i.e., $x^{2k+1} + y^{2k+1}$ is divisible by $x + y$.

Then $x^{2k+1} + y^{2k+1} = (x + y)\, p$, where p is the expression in x, y and is the quotient when $x^{2k+1} + y^{2k+1}$ is divided by $x + y$. ... (1)

We show that the statement is true for $n = k + 1$,

i.e., we show that $x^{2k+3} + y^{2k+3}$ is divisible by $x + y$.

From (1), we have $x^{2k+1} + y^{2k+1} = (x + y)\, p$

$$\therefore x^{2k+1} = (x+y)\ p - y^{2k+1}$$

$$\therefore x^{2k+1} . x^2 = (x+y)\ p\ x^2 - y^{2k+1} . x^2$$

$$\therefore x^{2k+3} = (x+y)\ p\ x^2 - y^{2k+1} . x^2$$

$$\therefore x^{2k+3} + y^{2k+3} = (x+y)\ p x^2 - y^{2k+1} . x^2 + y^{2k+3}$$

$$= (x+y)\ p x^2 - y^{2k+1}(x^2 - y^2)$$

$$= (x+y)\ p x^2 - y^{2k+1}(x+y)(x-y)$$

$$= (x+y)\left[p x^2 - y^{2k+1}(x-y)\right],$$

$\therefore x^{2k+3} + y^{2k+3}$ is divisible by $x + y$.

$\therefore$ The statement P(n) is true for $n = k + 1$.

$\therefore$ By the principle of mathematical induction, P(n) is true for all n.

i.e., $x^{2n+1} + y^{2n+1}$ is divisible by $x + y$, for all non-negative integers n.

i.e., $x^m + y^m$ is divisible by $x + y$, if m is an odd natural number.

Note : The above problem need not hold when m is an even natural number.

For example, if $m = 2$, $x = 1$, $y = 2$ then $x^2 + y^2 = 1^2 + 2^2 = 5$ is not divisible by $x + y = 1 + 2 = 3$.

3. Problem: *Show that* $49^n + 16n - 1$ *is divisible by 64 for all positive integers* n.

Solution: Let P(n) be the statement :

$49^n + 16n - 1$ is divisible by 64.

Since $49^1 + 16.1 - 1 = 64$ is divisible by 64, the statement is true for $n = 1$.

Assume that the statement P(n) is true for $n = k$, $k \geq 1$.

i.e., $49^k + 16k - 1$ is divisible by 64.

Then $49^k + 16k - 1 = 64t$, for some $t \in \mathbf{N}$. ... (1)

We show that the statement P(n) is true for $n = k + 1$,

i.e., we show that $49^{k+1} + 16(k+1) - 1$ is divisible by 64.

From (1), we have $49^k + 16k - 1 = 64t$

$\therefore\ 49^k = 64t - 16k + 1$

$\therefore\ 49^k.49 = (64t - 16k + 1).49$

$\therefore\ 49^{k+1} + 16\,(k+1) - 1 = (64t - 16k + 1).49 + 16\,(k+1) - 1$

$\therefore\ 49^{k+1} + 16\,(k+1) - 1 = 64\,(49\,t - 12k + 1),$

here $49t - 12\,k + 1$ is an integer.

$\therefore\ 49^{k+1} + 16\,(k+1) - 1$ is divisible by 64.

$\therefore$ The statement is true for $n = k + 1$.

$\therefore$ By the principle of mathematical induction, P(n) is true for all $n \in \mathbf{N}$,

i.e., $49^n + 16\,n - 1$ is divisible by 64, $\forall\ n \in \mathbf{N}$.

4. Problem : *Use mathematical induction to prove that* $2.4^{(2n+1)} + 3^{(3n+1)}$ *is divisible by* 11, $\forall\, n \in \mathbf{N}$.

Solution : Let P(n) be the statement :

$2.4^{(2n+1)} + 3^{(3n+1)}$ is divisible by 11.

Since $2.4^{(2.1+1)} + 3^{(3.1+1)} = 2.4^3 + 3^4 = 209 = 11 \times 19$ is divisible by 11, the statement P(n) is true for $n = 1$.

Assume that the statement P(n) is true for $n = k$, $k \geq 1$.

i.e., $2.4^{(2.k+1)} + 3^{(3.k+1)}$ is divisible by 11.

Then $2.4^{(2.k+1)} + 3^{(3.k+1)} = 11\,t$, for some integer t. ... (1)

We show that the statement P(n) is true for $n = k + 1$.

i.e., we show that $2.4^{2k+3} + 3^{3k+4}$ is divisible by 11.

From (1), we have $2.4^{(2k+1)} + 3^{(3k+1)} = 11t$

$$\therefore\ 2.4^{(2k+1)} = 11t - 3^{(3k+1)}$$

$$\therefore 2.\,4^{(2k+1)}\,.\,4^2 = \left(11t - 3^{(3k+1)}\right).\,4^2$$

$$2.\,4^{(2k+3)} + 3^{(3k+4)} = \left(11t - 3^{(3k+1)}\right)16 + 3^{(3k+4)}.$$

$$= 11t\,.\,16 - 3^{(3k+1)}.\,16 + 3^{(3k+4)}$$

$$= 11.t.16 + 3^{(3k+1)}\left[3^3 - 16\right]$$

$$= 11.t.16 + 3^{(3k+1)}\ (11)$$

$$= 11.\left[16t + 3^{(3k+1)}\right],$$

here $16t + 3^{(3k+1)}$ is an integer.

$\therefore$ $2.4^{(2k+3)} + 3^{(3k+4)}$ is divisible by 11.

$\therefore$ The statement P(n) is true for $n = k + 1$.

$\therefore$ By the principle of mathematical induction, P(n) is true for all $n \in \mathbf{N}$.

i.e., $2.\,4^{(2n+1)} + 3^{(3n+1)}$ is divisible by 11 for all $n \in \mathbf{N}$.

Note: While proving the statements using the principle of mathematical induction, the two steps : Basis of induction and Inductive hypothesis are important. Careless use of the principle of mathematical induction can lead to obviously absurd conclusions. There are statements that are true for many natural numbers but are not true for all of them as can be seen from the following examples.

2.3.2 Examples

(i) *The formula* $P(n) : n^2 - n + 41$ *gives a prime number for* $n = 1, 2, 3, ...,40$. *But P(41) =* 41^2 *is obivously divisible by 41. Therefore, it is not a prime number.*

(ii) *For* $n \in \mathbf{N}$, *let P(n) be the statement*

$$"1 + 3 + 5 + \cdots + (2n-1) = n^2 + (n-1)\,(n-2)\cdots(n-10)"$$

Then P(1), P(2), ... , P(10) are all true.

But P(11) is not true.

Exercise 2(a)

Using mathematical induction, prove each of the following statements, for all $n \in \mathbf{N}$.

1. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

2. $2.3 + 3.4 + 4.5 + \cdots$ upto n terms $= \dfrac{n(n^2 + 6n + 11)}{3}$.

3. $\dfrac{1}{1.3} + \dfrac{1}{3.5} + \dfrac{1}{5.7} + \cdots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$.

4. $4^3 + 8^3 + 12^3 + \cdots$ upto n terms $= 16n^2(n+1)^2$.

5. $a + (a+d) + (a+2d) + \cdots$ upto n terms $= \dfrac{n}{2}[2a + (n-1)d]$.

6. $a + ar + ar^2 + \cdots$ upto n terms $= \dfrac{a\,(r^n - 1)}{(r-1)}$, $r \neq 1$.

7. $2 + 7 + 12 + \cdots + (5n - 3) = \dfrac{n\,(5n-1)}{2}$.

8. $\left(1+\dfrac{3}{1}\right)\left(1+\dfrac{5}{4}\right)\left(1+\dfrac{7}{9}\right)\cdots\left(1+\dfrac{2n+1}{n^2}\right) = (n+1)^2$.

9. $(2n + 7) < (n + 3)^2$.

10. $1^2 + 2^2 + \ldots + n^2 > \dfrac{n^3}{3}$.

11. $4^n - 3n - 1$ is divisible by 9.

12. $3.5^{2n+1} + 2^{3n+1}$ is divisible by 17.

13. $1.2.3 + 2.3.4 + 3.4.5 + \cdots$ upto n terms $= \dfrac{n\,(n+1)\,(n+2)\,(n+3)}{4}$.

14. $\dfrac{1^3}{1} + \dfrac{1^3+2^3}{1+3} + \dfrac{1^3+2^3+3^3}{1+3+5} + \ldots$ upto n terms $= \dfrac{n}{24}\left[2n^2 + 9n + 13\right]$.

15. $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \cdots$ upto n terms $= \dfrac{n\,(n+1)^2\,(n+2)}{12}$.

Key Concepts

- Principle of finite mathematical induction:
  Let S be a subset of **N** such that
  (i) $1 \in S$
  (ii) For any $k \in \mathbf{N}$, $k \in S \Rightarrow k+1 \in S$.
  Then $S = \mathbf{N}$.

- Principle of complete mathematical induction:
  Let S be a subset of **N** such that
  (i) $1 \in S$
  (ii) For any $k \in \mathbf{N}$, $\{1, 2, 3, \ldots k\} \subseteq S \Rightarrow k+1 \in S$.
  Then $S = \mathbf{N}$.

- Steps to prove a statement using the principle of mathematical induction:
  (i) Basis of induction : Show that P(1) is true.
  (ii) Inductive hypothesis : For $k \geq 1$, assume that P(k) is true.
  (iii) Inductive step : Show that P(k + 1) is true on the basis of the inductive hypothesis.

Historical Note

Unlike other concepts and methods, proof by mathematical induction is not the invention of a particular individual at a particular moment. It is said that the principle of mathematical induction was known to the Phythagoreans.

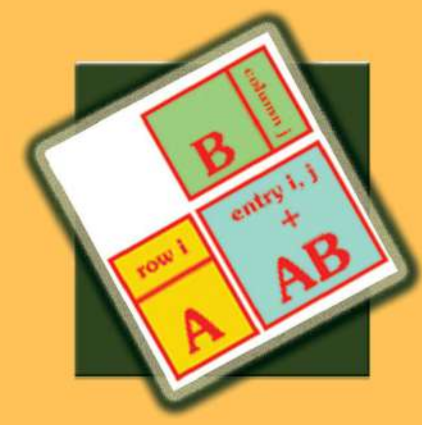
The French mathematician *Blaise Pascal* (1623 - 1662) is credited with the origin of the principle of mathematical induction.

The name 'induction' was used by the English mathematician *John Wallis* (1616-1703).

Later the principle was employed to provide a proof of the binomial theorem.

De Morgan (1806 - 1871) had many accomplishments in the field of mathematics on many different subjects. He was the first person to define 'mathematical induction' and developed De Morgan's rule in set theory and wrote a treatise on formal logic.

Giuseppe Peano (1858 - 1932) undertook the task of deducing the properties of natural numbers from a set of explicitly stated assumptions, now known as Peano's axioms. The principle of mathematical induction is a restatement of one of Peano's axioms.



Chapter 3

Matrices

"The search for truth is more precious than its possession"

- Albert Einstein

Introduction

We have learnt about Matrices and their determinants in high school classes. An arrangement of numbers in a rectangular array comprising of rows and columns is known as a matrix. $m \times n$ (read as m by n), where m is the number of rows and n is the number of columns, is known as the order of the matrix. In high school classes our study was limited to 2×2 matrices.

In this chapter we deal with higher order matrices in general and 3×3 matrices in particular. For the sake of completeness, we shall start with defining a matrix etc... and go on to extend our study of the algebra of matrices and then use the theory to find the solutions of simultaneous linear equations.

Arthur Cayley

(1821 - 1895)

__Arthur Cayley__ was a British mathematician. Cayley worked as a lawyer for 14 years. While he was a lawyer he published about 250 research papers in mathematics, and later, while working as Sadleirian Professor at Cambridge, published another 650. It was Cayley who first introduced matrix multiplication. He was consequently able to prove the Cayley-Hamilton theorem - that every square matrix is a root of its own characteristic polynomial.

3.1 Types of matrices

In this section, we define a matrix, its order and various types of matrices.

3.1.1 Definition (Matrix)

An ordered rectangular array of elements is called a ***matrix.***

We confine our discussion to matrices whose elements are real or complex numbers; or real or complex valued functions. Matrices are generally enclosed by brackets.

We denote matrices by capital letters A,B, C...

The following are some examples of matrices.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & -6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$$

$$C = \begin{bmatrix} x+1 & x^2-1 & 3 \\ -3 & 2 & \sin x \\ 7+\sin x & 4 & 3+\sin 2x \end{bmatrix} \begin{matrix} \rightarrow 1^{st} \text{ row} \\ \rightarrow 2^{nd} \text{ row} \\ \rightarrow 3^{rd} \text{ row} \end{matrix}$$

$$\begin{matrix} \downarrow & \downarrow & \downarrow \\ 1^{st} \text{ column} & 2^{nd} \text{ column} & 3^{rd} \text{ column} \end{matrix}$$

In the above examples, the horizontal lines of elements are said to constitute **the rows** of the matrix and the vertical lines of elements are said to constitute **the columns** of the matrix. Thus A has 2 rows and 3 columns, B has 2 rows and 2 columns, while C has 3 rows and 3 columns.

3.1.2 Definition (Order of Matrix)

A matrix having m rows and n columns is said to be of ***order m × n,*** *read as m cross n or m by n.*

In the above examples, A is of order 2×3, B is of order 2×2 and C is of order 3×3.

In general, a matrix having *m* rows and *n* columns is represented as follows.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & & & & \\ \cdots & \cdots & & & \cdots & \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

In the above matrix every element is specified by its position in terms of the row and column in which the element is present. The first and second suffices of an element indicate respectively the row and column in which the element is present. For example a_{23} is the element present in the second row and the third column.

In compact form the above matrix is denoted by
$A = [a_{ij}]_{m \times n}$ where $1 \le i \le m$ and $1 \le j \le n$.

Throughout this chapter, we generally consider matrices of order $m \times n$, where $m \in \{1, 2, 3\}$ and $n \in \{1, 2, 3, 4\}$.

3.1.3 Types of matrices

1. Square matrix

A matrix in which the number of rows is equal to the number of columns, is called a ***square matrix***.

$A = [a_{ij}]_{m \times n}$ is a square matrix if $m = n$. In this case we say that A is a square matrix of order m. For example,

$[2]$ is a square matrix of order 1.

$\begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix}$ is a square matrix of order 2,

and $\begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 2 \\ 7 & 6 & 9 \end{bmatrix}$ is a square matrix of order 3.

If $A = [a_{ij}]$ is a square matrix of order n, the elements $a_{11}, a_{22}, \ldots, a_{nn}$ are said to constitute its ***Principal diagonal*** or simply the ***diagonal***. Hence a_{ij} is an element of the diagonal or non-diagonal according as $i = j$ or $i \ne j$.

The sum of the elements of the diagonal of a square matrix A is called the ***trace*** of A and is denoted by Tr (A).

If $A = [a_{ij}]$ is a square matrix of order n, then $\text{Tr}(A) = \sum_{i=1}^{n} a_{ii}$.

For example, if $A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 2 \\ 7 & 6 & 9 \end{bmatrix}$, then $\text{Tr}(A) = 2 + (-1) + 9 = 10$.

2. Diagonal matrix

If each non-diagonal element of a square matrix is equal to zero, then the matrix is called a ***diagonal matrix***.

For example, $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are diagonal matrices.

If $A = [a_{ij}]_{n \times n}$ is a diagonal matrix, it is sometimes denoted as diag $[a_{11}, a_{22}, ..., a_{nn}]$.

3. Scalar matrix

If each non-diagonal element of a square matrix is zero and all diagonal elements are equal to each other, then it is called a ***scalar matrix*****.**

For example, $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$,

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ are all scalar matrices.

4. Unit (Identity) matrix

If each non-diagonal element of a square matrix is equal to zero and each diagonal element is equal to 1, then that matrix is called a ***Unit matrix*** ***or*** ***Identity matrix*****.**

We denote the unit matrix of order n by I_n, or simply by I, when there is no ambiguity about the order.

For example, $I_1 = [1]$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are unit matrices.

$[a_{ij}]_{n \times n}$ is a unit matrix
$\Leftrightarrow a_{ij} = 1$ if $i = j$ and $a_{ij} = 0$ if $i \neq j$

5. Null matrix or Zero matrix

If each element of a matrix is zero, then it is called a ***Null matrix*** ***or*** ***Zero matrix*****.** *It is denoted by* $O_{m \times n}$ *or simply by* O.

For example, $O_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $O_{3 \times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ are null matrices.

6. Row matrix and Column matrix

A matrix with only one row is called a ***Row matrix*** ***(or row vector)*** *and a matrix with only one column is called a* ***Column matrix*** ***(or column vector)*****.**

For example, $[1 \quad 3 \quad -2]$ is a row matrix (order 1×3),

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a column matrix (order 2×1).

7. Triangular matrices

A square matrix $A = [a_{ij}]$ *is said to be* ***Upper Triangular*** *if* $a_{ij} = 0$ *for all* $i > j$.

A is said to be ***Lower Triangular*** *if* $a_{ij} = 0$ *for all* $i < j$.

For example, $\begin{bmatrix} 2 & -4 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -3 & 1 \\ 0 & 4 \end{bmatrix}$ are upper triangular matrices while

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ are lower triangular matrices.

Observe that I_3 and O_3 are both upper and lower triangular matrices.

> $A = [a_{ij}]_{n \times n}$, is
>
> Upper Triangular if $a_{ij} = 0$ for *all* $i > j$.
>
> Lower Triangular if $a_{ij} = 0$ for *all* $i < j$

3.1.4 Definition (Equality of matrices)

Matrices A and B are said to be equal if A and B are of the same order and the corresponding elements of A and B are the same.

Thus $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

are equal if $a_{ij} = b_{ij}$ for $i = 1, 2$ and $j = 1, 2, 3$.

3.1.5 Definition (Sum of two matrices)

Let A and B be matrices of the same order. Then the sum of A and B, denoted by A + B, is defined as the matrix of the same order in which each element is the sum of the corresponding elements of A and B.

> If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$,
>
> then $A + B = [c_{ij}]_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij}$

For example, if $A = \begin{bmatrix} 3 & 2 & -1 \\ 4 & -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & 7 \\ 3 & 2 & -1 \end{bmatrix}$ then

$$A + B = \begin{bmatrix} 3+1 & 2+(-2) & -1+7 \\ 4+3 & -3+2 & 1+(-1) \end{bmatrix} = \begin{bmatrix} 4 & 0 & 6 \\ 7 & -1 & 0 \end{bmatrix}.$$

3.1.6 Properties of Addition of matrices

Let $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$ be matrices of the same order. Then the addition of matrices satisfies the following properties :

(i) *Commutative Property*

$$A + B = B + A$$

Now
$$\begin{aligned} A + B &= [a_{ij}] + [b_{ij}] \\ &= [a_{ij} + b_{ij}] \\ &= [b_{ij} + a_{ij}] \\ &= [b_{ij}] + [a_{ij}] \\ &= B + A \end{aligned}$$

Addition of matrices is commutative.
i.e., $A + B = B + A$

(ii) *Associative Property*

$$A + (B + C) = (A + B) + C$$

Now
$$\begin{aligned} (A + B) + C &= \left([a_{ij}] + [b_{ij}]\right) + [c_{ij}] \\ &= [a_{ij} + b_{ij}] + [c_{ij}] \\ &= [(a_{ij} + b_{ij}) + c_{ij}] \\ &= [a_{ij} + (b_{ij} + c_{ij})] \quad \text{(why?)} \\ &= [a_{ij}] + [b_{ij} + c_{ij}] \\ &= [a_{ij}] + \left([b_{ij}] + [c_{ij}]\right) \\ &= A + (B + C) \end{aligned}$$

Addition of matrices obeys Associative Property
i.e., $A + (B + C) = (A + B) + C$

(iii) *Additive identity*

If A is a $m \times n$ matrix and O is the $(m \times n)$ null matrix,
$A + O = O + A = A$. We call O the additive indentity in the set of all $m \times n$ matrices.

(iv) *Additive inverse*

If A is an $(m \times n)$ matrix then there is a unique $m \times n$ matrix B such that
$A + B = B + A = O$, O being the $m \times n$ null matrix.

This B is denoted by $-A$ and is called the additive inverse of A. Infact if $A = [a_{ij}]$, then $B = [-a_{ij}]$.

3.2 Scalar multiple of a matrix and multiplication of matrices

This section is devoted to the study of multiplication of a matrix (i) by a scalar and (ii) by a matrix. We also study the properties of multiplication.

3.2.1 Definition (Scalar multiple of a matrix)

Let A be a matrix of order $m \times n$ and k be a scalar (i.e., real or complex number). Then the $m \times n$ matrix obtained by multiplying each element of A by k is called a scalar multiple of A and is denoted by k A.

If $A = [a_{ij}]_{m\times n}$ then $kA = [ka_{ij}]_{m\times n}$

For example if $k = 2$ and $A = \begin{bmatrix} 3 & 2 & -1 \\ 4 & -3 & 1 \end{bmatrix}$ then

$$kA = 2A = \begin{bmatrix} 2\times 3 & 2\times 2 & 2\times(-1) \\ 2\times 4 & 2\times(-3) & 2\times 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 & -2 \\ 8 & -6 & 2 \end{bmatrix}.$$

3.2.2 Note

$(-1)A = -A$ because $A + (-1)A = O$.

3.2.3 Properties of scalar multiplication of a matrix

Let A and B be matrices of the same order and α, β be scalars. Then

(i) $\alpha(\beta A) = (\alpha\beta)A = \beta(\alpha A)$

(ii) $(\alpha+\beta)A = \alpha A + \beta A$

(iii) $\alpha(A+B) = \alpha A + \alpha B$

(iv) $\alpha O = O$

(v) $0A = O$

Consider (ii) Let $A = [a_{ij}]_{m\,x\,n}$

$$\begin{aligned}(\alpha+\beta)A &= (\alpha+\beta)[a_{ij}] \\ &= [(\alpha+\beta)a_{ij}] \quad \text{..... by definition 3.2.1} \\ &= [\alpha a_{ij} + \beta a_{ij}] \quad \text{..... by distributive law of numbers} \\ &= [\alpha a_{ij}] + [\beta a_{ij}] \\ &= \alpha[a_{ij}] + \beta[a_{ij}] \\ &= \alpha A + \beta A\end{aligned}$$

Verification of properties (i), (iii), (iv) and (v) is left to the student as an exercise.

3.2.4 Solved Problems

1. Problem: *If* $A = \begin{bmatrix} 2 & 3 & -1 \\ 7 & 8 & 5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & -1 \end{bmatrix}$ *then find A + B.*

Solution : A + B is defined since A and B are of same order.

$$A+B = \begin{bmatrix} 2 & 3 & -1 \\ 7 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+1 & 3+0 & -1+1 \\ 7+2 & 8-4 & 5-1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 0 \\ 9 & 4 & 4 \end{bmatrix}.$$

2. Problem : *If* $\begin{bmatrix} x-1 & 2 & y-5 \\ z & 0 & 2 \\ 1 & -1 & 1+a \end{bmatrix} = \begin{bmatrix} 1-x & 2 & -y \\ 2 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ *then find the values of x, y, z and a.*

Solution : From the equality of matrices

$$x-1 = 1-x;\ y-5 = -y;\ z = 2;\ 1+a = 1.$$

Hence $x = 1;\quad y = \frac{5}{2};\quad z = 2;\quad a = 0$.

3. Problem : *Find the trace of* A *if* $A = \begin{bmatrix} 1 & 2 & -\frac{1}{2} \\ 0 & -1 & 2 \\ -\frac{1}{2} & 2 & 1 \end{bmatrix}$.

Solution : The elements of the Principal diagnonal of A are 1, -1, 1. Hence the trace of A is $1 + (-1) + 1 = 1$.

4. Problem : *If* $A = \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}$ *then find* $-5A$.

Solution: By the definition of scalar multiplication of matrix

$$-5A = (-5)\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} (-5)4 & (-5)(-5) \\ (-5)(-2) & (-5)3 \end{bmatrix} = \begin{bmatrix} -20 & 25 \\ 10 & -15 \end{bmatrix}.$$

5. Problem: *Find the additive inverse of A where*

$$A = \begin{bmatrix} i & 0 & 1 \\ 0 & -i & 2 \\ -1 & 1 & 5 \end{bmatrix}.$$

Solution : The additive inverse of A is $-A = (-1)A$.

Hence the additive inverse of the given matrix

$$-A = (-1)\begin{bmatrix} i & 0 & 1 \\ 0 & -i & 2 \\ -1 & 1 & 5 \end{bmatrix} = \begin{bmatrix} -i & 0 & -1 \\ 0 & i & -2 \\ 1 & -1 & -5 \end{bmatrix}.$$

6. Problem : *If* $A = \begin{bmatrix} 2 & 3 & 1 \\ 6 & -1 & 5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ *then find the matrix* X *such that* $A + B - X = 0$. *What is the order of the matrix* X ?

Solution: A and B are matrices of the same order 2×3. If $A + B - X$ is to be defined, the order of X also must be 2×3.

$$A + B - X = O \Leftrightarrow X = A + B$$

$$\therefore X = \begin{bmatrix} 2 & 3 & 1 \\ 6 & -1 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 & 0 \\ 6 & -2 & 8 \end{bmatrix}.$$

7. Problem : *If* $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 3 \end{bmatrix}$ *then find* $A-B$ *and* $4B - 3A$.

Solution: $A - B = \begin{bmatrix} 0-1 & 1-(-2) & 2-0 \\ 2-0 & 3-1 & 4-(-1) \\ 4-(-1) & 5-0 & 6-3 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 2 & 5 \\ 5 & 5 & 3 \end{bmatrix}$

$$4B - 3A = 4\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -8 & 0 \\ 0 & 4 & -4 \\ -4 & 0 & 12 \end{bmatrix} - \begin{bmatrix} 0 & 3 & 6 \\ 6 & 9 & 12 \\ 12 & 15 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -11 & -6 \\ -6 & -5 & -16 \\ -16 & -15 & -6 \end{bmatrix}.$$

8. Problem : *If* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}$ *and* $2X + A = B$ *then find* X.

Solution : $2X + A = B \Rightarrow 2X = B - A$

$$= \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 6 \\ 4 & -2 \end{bmatrix}$$

$$\text{Hence } X = \frac{1}{2}\begin{bmatrix} 2 & 6 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}.$$

9. Problem : *Two factories I and II produce three varieties of pens namely, Gel, Ball and Ink pens. The sale in rupees of these varieties of pens by both the factories in the month of September and October in a year are given by the following matrices A and B.*

September sales (in Rupees)

$$A = \begin{array}{c} \begin{array}{ccc} \textit{Gel} & \textit{Ball} & \textit{Ink} \end{array} \\ \begin{bmatrix} 1000 & 2000 & 3000 \\ 5000 & 3000 & 1000 \end{bmatrix} \end{array} \begin{array}{l} \text{Factory I} \\ \text{Factory II} \end{array}$$

October sales (in Rupees)

$$B = \begin{array}{c} \begin{array}{ccc} \textit{Gel} & \textit{Ball} & \textit{Ink} \end{array} \\ \begin{bmatrix} 500 & 1000 & 600 \\ 2000 & 1000 & 1000 \end{bmatrix} \end{array} \begin{array}{l} \text{Factory I} \\ \text{Factory II} \end{array}$$

(i) *Find the combined sales in September and October for each factory in each variety.*

(ii) *Find the decrease in sales from September to October.*

Solution : (i) Combined sales in September and October for each factory in each variety is given by

$$A+B = \begin{array}{c} \begin{array}{ccc} \text{Gel} & \text{Ball} & \text{Ink} \end{array} \\ \begin{bmatrix} 1500 & 3000 & 3600 \\ 7000 & 4000 & 2000 \end{bmatrix} \end{array} \begin{array}{l} \text{Factory I} \\ \text{Factory II} \end{array}$$

(ii) Decrease in sales from September to October is given by

$$A - B = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \text{Gel} \quad \text{Ball} \quad \text{Ink} \\ \begin{bmatrix} 500 & 1000 & 2400 \\ 3000 & 2000 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Factory I} \\ \text{Factory II} \end{array}$$

10. Problem : *Construct a 3 × 2 matrix whose elements are defined by* $a_{ij} = \frac{1}{2}|i-3j|$.

Solution : In general a 3×2 matrix is given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Now $a_{ij} = \frac{1}{2}|i-3j| \quad i = 1, 2, 3$ and $j = 1, 2.$

$$a_{11} = \frac{1}{2}|1-(3\times1)| = 1 \qquad a_{12} = \frac{1}{2}|1-(3\times2)| = \frac{5}{2}$$

$$a_{21} = \frac{1}{2}|2-(3\times1)| = \frac{1}{2} \qquad a_{22} = \frac{1}{2}|2-(3\times2)| = 2$$

$$a_{31} = \frac{1}{2}|3-(3\times1)| = 0 \qquad a_{32} = \frac{1}{2}|3-(3\times2)| = \frac{3}{2}$$

$$\therefore \quad A = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{1}{2} & 2 \\ 0 & \frac{3}{2} \end{bmatrix}$$

Exercise 3(a)

I. **1.** Write the following as a single matrix.

(i) $\begin{bmatrix} 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 9 & 0 \\ 1 & 8 & -2 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 2 \\ 7 & 1 & 4 \end{bmatrix}$

(iv) $\begin{bmatrix} -1 & 2 \\ 1 & -2 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -2 & 1 \end{bmatrix}$

2. If $A = \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ and $A + B = X$ then find the values of x_1, x_2, x_3 and x_4.

3. If $A = \begin{bmatrix} -1 & -2 & 3 \\ 1 & 2 & 4 \\ 2 & -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 5 \\ 0 & -2 & 2 \\ 1 & 2 & -3 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

then find $A + B + C$.

4. If $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & -1 & 0 \\ 2 & 1 & 3 \\ 4 & -1 & 2 \end{bmatrix}$ and $X = A + B$ then find X.

5. If $\begin{bmatrix} x-3 & 2y-8 \\ z+2 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -2 & a-4 \end{bmatrix}$ then find the values of x, y, z and a.

II. 1. If $\begin{bmatrix} x-1 & 2 & 5-y \\ 0 & z-1 & 7 \\ 1 & 0 & a-5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 1 & 0 & 0 \end{bmatrix}$ then find the values of x, y, z and a.

2. Find the trace of $\begin{bmatrix} 1 & 3 & -5 \\ 2 & -1 & 5 \\ 2 & 0 & 1 \end{bmatrix}$.

3. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ find $B - A$ and $4A - 5B$.

4. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ find $3B - 2A$.

3.2.5 Multiplication of matrices

We say that matrices A *and* B *are* ***conformable for multiplication*** *in that order (giving the product* AB) *if the number of columns of* A *is equal to the number of rows of* B.

3.2.6 Definition (Product of two matrices)

Let $A = [a_{ik}]_{m\times n}$ *and* $B = [b_{kj}]_{n\times p}$, *be two matrices. Then the matrix* $C = [c_{ij}]_{m\times p}$ *where* $c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$ *is called the product of A and B and is denoted by AB.*

Observe that when the orders of A and B are $m \times n$ and $n \times p$, the order of the product matrix AB is $m \times p$. Every element of AB is in the form of a sum of products of certain elements of A and of B. For example, in $C = AB = [c_{ij}]_{m\times p}$

$$c_{23} = \sum_{k=1}^{n} a_{2k}\, b_{k3} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} + ... + a_{2n}b_{n3}$$

= the sum of the products of the elements of second row of A with the corresponding elements of the 3rd column of B

A useful method to understand and to remember matrix multiplication is illustrated in the following example.

$$\text{Let } A_{2\times 3} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix} \text{ and } B_{3\times 4} = \begin{bmatrix} 1 & 3 & -4 & 2 \\ -1 & 0 & 3 & 5 \\ 0 & 4 & 7 & -6 \end{bmatrix}$$

Let the rows of A be R_1, R_2 and the columns of B be C_1, C_2, C_3, C_4. When $A_{2\times 3}$ is multiplied with $B_{3\times 4}$, the order of the product matrix C = AB is 2×4.

$$\therefore \text{ Let } C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{bmatrix}.$$

$$\text{Then } C = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 & R_1C_4 \\ R_2C_1 & R_2C_2 & R_2C_3 & R_2C_4 \end{bmatrix}.$$

$c_{11} = R_1C_1$ = sum of the products of the 1st row elements of A with the corresponding elements of the 1st column of B.
$= 2(1) + 3(-1) + 1(0) = -1.$

$c_{12} = R_1C_2$ = sum of the products of the 1st row elements of A with the corresponding elements of the 2nd column of B
$= 2(3) + 3(0) + 1(4) = 10.$

3.2.7 Examples

1. Example

Consider the matrices $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix}$.

Clearly A, B as well as B, A are conformable.

Further $AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2(0)+3(-1) & 2(4)+3(2) \\ 1(0)+2(-1) & 1(4)+2(2) \end{bmatrix} = \begin{bmatrix} -3 & 14 \\ -2 & 8 \end{bmatrix}$.

Now $BA = \begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0(2)+4(1) & 0(3)+4(2) \\ -1(2)+2(1) & -1(3)+2(2) \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 0 & 1 \end{bmatrix}$.

Hence the products AB and BA are not necessarily equal.

2. Example

A certain bookshop has 10 dozen chemistry books, 8 dozen physics books, 10 dozen economics books. Their selling prices are Rs. 80, Rs. 60 and Rs. 40 each respectively. Using matrix algebra, find the total value of the books in the shop.

Solution: Number of books

Chemistry Physics Economics

$$A = \begin{bmatrix} 10\times 12 & 8\times 12 & 10\times 12 \\ =120 & =96 & =120 \end{bmatrix}$$

Selling price (in rupees)

$$B = \begin{bmatrix} 80 \\ 60 \\ 40 \end{bmatrix} \begin{matrix} \text{Chemistry} \\ \text{Physics} \\ \text{Economics} \end{matrix}$$

Total value of the books in the shop.

$$AB = [120 \quad 96 \quad 120]\begin{bmatrix} 80 \\ 60 \\ 40 \end{bmatrix}$$

$$= [120 \times 80 + 96 \times 60 + 120 \times 40]$$

$$= [9600 + 5760 + 4800]$$

$$= [20160] \text{ (in rupees).}$$

3.2.8 Note

Matrix multiplication is not commutative. If A and B are matrices conformable for multiplication, AB exists, but BA may not exist; even if BA exists, AB and BA may not have the same order and even if they have the same order they may not be equal.

1. If the orders of A and B are 2×3 and 3×4 respectively then the order of AB is 2×4, but BA does not exist. (The number of columns of B is not equal to the number of rows of A, that is B and A are not conformable for multiplication).

2. If the orders A and B are 2×3 and 3×2 respectively, then the order of AB is 2×2, while the order of BA is 3×3. Hence AB and BA can not be equal.

3. For the matrices A and B of example 1, 3.2.7, AB and BA have the same order but $AB \neq BA$.

This does not mean however, that $AB \neq BA$ for every pair of matrices A, B for which AB and BA are defined and are of same order.

For instance, $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ then $AB = BA = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$.

Verify whether every pair of diagonal matrices of same order commute or not!

Also, verify by an example whether a pair of square matrices of same order, whose product is a scalar matrix, commute or not!

3.2.9 Note

Let $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$ then $AB = BA = O$.

We know that in case of real numbers a, b if $ab = 0$ then $a = 0$ or $b = 0$. But in matrices, the product of two non-zero matrices could be a zero matrix, as seen from the above example.

3.2.10 Note

If $AB = AC$ and $A \neq O$, then it is not necessary that $B = C$.

For example, if $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 5 & 6 \end{bmatrix}$

we have $AB = O = AC$, but $B \neq C$.

3.2.11 Properties of multiplication of matrices

Multiplication of matrices possesses the following properties, which we state without proof.

1. The Associative Law

For any three matrices A, B and C, we have $(AB)C = A(BC)$ in the sense that whenever one side of the equality is defined, then the otherside is also defined and the equality holds.

2. The Distributive Law

For any three matrices A, B and C, we have

(i) $A(B + C) = AB + AC$ (Left Distributive Law)

(ii) $(A+B)C = AC + BC$ (Right Distributive Law)

in the sense that whenever one side of the equation is defined, then the otherside is also defined and the equality holds.

3. Existence of multiplicative identity

If I is the identity matrix of order n, then for every square matrix A of order n

$$IA = AI = A.$$

3.2.12 Note

(i) For any square matrix A, we denote $A \cdot A$ by A^2. In general, for any positive integer $n, n > 1$, the product A. A. A ... A (taken n times) is denoted by A^n.

(ii) If A and B are matrices of orders $m \times n$ and $n \times p$ respectively and α, β are scalars, then

$$(\alpha A).(\beta B) = \alpha\beta(AB) = ((\alpha\beta)A)B = A.((\alpha\beta)B).$$

(iii) If α is a scalar, A is a square matrix and n is a positive integer, then

$$(\alpha A)^n = \alpha^n A^n \text{ and } \alpha A = (\alpha I)A.$$

We now verify all the properties of multiplication, in the following solved problems.

3.2.13 Solved Problems

1. Problem

If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$ *then find AB and BA.*

Solution : The number of columns of $A = 3 =$ the number of rows of B. Hence AB is defined and

$$AB = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1+1.(-1)+2.2 & 0.(-2)+1.0+2.(-1) \\ 1.1+2.(-1)+3.2 & 1.(-2)+2.0+3.(-1) \\ 2.1+3.(-1)+4.2 & 2.(-2)+3.0+4.(-1) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$$

Since the number of columns of B is not equal to the number of rows of A, BA is not defined.

2. Problem: *If* $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ *then examine whether A and B commute with respect to multiplication of matrices.*

Solution: Both A and B are square matrices of order 3. Hence both AB and BA are defined and are matrices of order 3.

$$AB = \begin{bmatrix} 1.1+(-2).0+3.1 & 1.0+(-2).1+3.2 & 1.2+(-2).2+3.0 \\ 2.1+3.0+(-1).1 & 2.0+3.1+(-1).2 & 2.2+3.2+(-1).0 \\ (-3).1+1.0+2.1 & -3.0+1.1+2.2 & -3.2+1.2+2.0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & -2 \\ 1 & 1 & 10 \\ -1 & 5 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 7 \\ -4 & 5 & 3 \\ 5 & 4 & 1 \end{bmatrix}$$

which shows that $AB \neq BA$.

Therefore A and B do not commute with respect to multiplication of matrices.

3. Problem : *If* $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ *then show that* $A^2 = -I$.

Solution : $A^2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -I.$$

4. Problem: *If* $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ *then show that for all the postive integers n,*

$$A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}.$$

Solution : We solve this problem by using the principle of mathematical induction.

Consider the statement $P(n)$: $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$

Since $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, $P(n)$ is true for $n = 1$.

Suppose that the given statement $P(n)$ is true for $n = k$, $k \geq 1$.

$$\text{Then } A^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$$

$$\text{Consider } A^{k+1} = A^k.A = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos k\theta.\cos\theta - \sin k\theta.\sin\theta & \cos k\theta.\sin\theta + \sin k\theta.\cos\theta \\ -\sin k\theta.\cos\theta - .\cos k\theta\sin\theta & -\sin k\theta.\sin\theta + \cos k\theta.\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(k\theta+\theta) & \sin(k\theta+\theta) \\ -\sin(k\theta+\theta) & \cos(k\theta+\theta) \end{bmatrix} \begin{array}{l} (\because \cos(A+B) = \cos A.\cos B - \sin A.\sin B; \\ \quad \sin(A+B) = \sin A.\cos B + \cos A.\sin B\ ) \end{array}$$

$$= \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Therefore P(n) is true for $n = k+1$.

Hence, by mathematical induction, P(n) is true for all positive integral values of n.

5. Problem : *If* $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ *then show that* $A^2 - 4A - 5I = O$.

Solution : $A^2 = A\,.\,A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$

$$4A = 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$5I = 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Hence

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = O.$$

Exercise 3(b)

I. 1. Find the following products wherever possible

(i) $\begin{bmatrix} -1 & 4 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 1 & 4 \\ 6 & -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & -2 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 2 & 5 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} -2 & -3 & 4 \\ 2 & 2 & -3 \\ 1 & 2 & -2 \end{bmatrix}$

(v) $\begin{bmatrix} 3 & 4 & 9 \\ 0 & -1 & 5 \\ 2 & 6 & 12 \end{bmatrix}\begin{bmatrix} 13 & -2 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

(vi) $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 4 \\ 6 & -2 & 3 \end{bmatrix}$

(vii) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(viii) $\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$

2. If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$, do AB and BA exist? If they exist, find them. Do A and B commute with respect to multiplication ?

3. Find A^2, where $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$.

4. If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, find A^2.

5. If $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ and I is the unit matrix of order 2, then show that

(i) $A^2 = B^2 = C^2 = -I$

(ii) $AB = -BA = -C$.

6. If $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$, find AB. Find BA, if it exists.

7. If $A = \begin{bmatrix} 2 & 4 \\ -1 & k \end{bmatrix}$ and $A^2 = O$, then find the value of k.

II. 1. If $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then find A^4.

2. If $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$, then find A^3.

3. If $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ then find $A^3 - 3A^2 - A - 3I$, where I is unit matrix of order 3.

4. If $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then show that $(aI + bE)^3 = a^3I + 3a^2bE$, where I is unit matrix of order 2.

III. 1. If $A = \text{diag}[a_1, a_2, a_3]$, then for any integer $n \geq 1$ show that $A^n = \text{diag}[a_1^n, a_2^n, a_3^n]$.

2. If $\theta - \phi = \dfrac{\pi}{2}$, then show that

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix} = O.$$

3. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ then show that $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ for any integer $n \geq 1$, by using mathematical induction.

4. Give examples of two square matrices A and B of the same order for which $AB = O$, but $BA \neq O$.

5. A trust fund has to invest Rs. 30,000 in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide Rs. 30,000 among the two types of bonds, if the trust fund must obtain an annual total interest of (a) Rs. 1800 (b) Rs. 2000.

3.3 Transpose of a matrix

In this section we define the Transpose of a martrix and study its properties. We also define symmetric and skew symmetric matrices.

3.3.1 Definition (Transpose of a matrix)

If $A = [a_{ij}]$ is an $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the ***transpose of A****. Transpose of the matrix A is denoted by A' or A^T. In other words, if $A = [a_{ij}]_{m\times n}$, then $A' = [a_{ji}]_{n\times m}$.*

For example if

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \\ 0 & \sqrt{7} \end{bmatrix} \quad \text{then} \quad A' = \begin{bmatrix} 3 & 4 & 0 \\ 2 & 1 & \sqrt{7} \end{bmatrix}.$$

3.3.2 Properties of transpose of matrices

We now state the following properties of transpose of matrices without proof. These may be verified by taking suitable examples.

For any two matrices A, B of suitable orders, we have

(i) $(A')' = A$ (ii) $(kA)' = kA'$

(iii) $(A+B)' = A' + B'$ (iv) $(AB)' = B'A'$

3.3.3 Example

If $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}$ *and* $B = \begin{bmatrix} -3 & 4 & 0 \\ 4 & -2 & -1 \end{bmatrix}$

Verify that (i) $(A')' = A$ (ii) $(A+B)' = A' + B'$ (iii) $(5B)' = 5(B)'$

Solution

(i) We have $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}$

$$\Rightarrow \quad A' = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$$\Rightarrow (A')' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix} = A.$$

(ii) $$A+B = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix} + \begin{bmatrix} -3 & 4 & 0 \\ 4 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 8 & 7 \\ 6 & 3 & 7 \end{bmatrix}$$

$$\therefore\ (A+B)' = \begin{bmatrix} -2 & 6 \\ 8 & 3 \\ 7 & 7 \end{bmatrix}$$

$$A' + B' = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} + \begin{bmatrix} -3 & 4 \\ 4 & -2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ 8 & 3 \\ 7 & 7 \end{bmatrix} = (A+B)'.$$

(iii) We have $$5B = 5\begin{bmatrix} -3 & 4 & 0 \\ 4 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & 20 & 0 \\ 20 & -10 & -5 \end{bmatrix}$$

$$(5B)' = \begin{bmatrix} -15 & 20 \\ 20 & -10 \\ 0 & -5 \end{bmatrix}$$

also $$5B' = \begin{bmatrix} -15 & 20 \\ 20 & -10 \\ 0 & -5 \end{bmatrix}$$

Thus $(5B)' = 5B'$.

3.3.4 Example

If $A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & -2 \\ -3 & 0 \\ 5 & 4 \end{bmatrix}$ *then verify that* $(AB)' = B'A'$.

Solution

We have $AB = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -3 & 0 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 15 & 4 \\ -28 & -18 \end{bmatrix}$

$\therefore (AB)' = \begin{bmatrix} 15 & -28 \\ 4 & -18 \end{bmatrix}$

Now $A' = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 2 & -4 \end{bmatrix}$ and $B' = \begin{bmatrix} 1 & -3 & 5 \\ -2 & 0 & 4 \end{bmatrix}$

$\therefore B'A' = \begin{bmatrix} 1 & -3 & 5 \\ -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 15 & -28 \\ 4 & -18 \end{bmatrix}$

Hence $(AB)' = B'A'$.

3.3.5 Definition (Symmetric matrix)

A square matrix A is said to be ***symmetric*** *if* $A' = A$.

3.3.6 Note

(i) The zero matrix $O_{n\times n}$, any diagonal matrix and the unit matrix $I_{n\times n}$ are symmetric.

(ii) If A is a symmetric matrix, then the $(i, j)^{th}$ element of A is the same as the $(j, i)^{th}$ element of A.

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -3 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

Observe that $a_{12} = a_{21} = 2$, $a_{13} = a_{31} = 0$ and $a_{23} = a_{32} = -1$. So A is symmetric.

(iii) If A is a square matrix, then $A + A'$ is a symmetric matrix.

3.3.7 Definition (Skew-symmetric matrix)

A square matrix A is said to be ***skew-symmetric*** *if* $A' = -A$.

For example, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -4 & 0 \end{bmatrix}$ are skew - symmetric matrices.

3.3.8 Note

(i) The zero matrix $O_{n\times n}$ is skew-symmetric.

(ii) If A is a **skew-symmetric** matrix, then the $(i, j)^{th}$ element of A is the same as the negative of the $(j,i)^{th}$ element of A.

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -4 & 0 \end{bmatrix}$$

Observe that $a_{12} = 1 = -a_{21}$, $a_{13} = -2 = -a_{31}$ and $a_{23} = 4 = -a_{32}$,

since the diagonal elements a_{11}, a_{22} and a_{33} do not change while transposing the given matrix, if $A = [a_{ij}]_{n\times n}$ is a skew symmetric matrix, then $a_{ii} = -a_{ii}$ so that $a_{ii} = 0 (i = 1, 2, ...n)$.

(iii) If A is a square matrix, then $A - A'$ is a skew-symmetric matrix.

(iv) If A is a symmetric (or skew-symmetric) matrix, then kA is also symmetric (or skew-symmetric) for any scalar k.

3.3.9 Solved Problems

1. Problem : *If* $A = \begin{bmatrix} -2 & 1 & 0 \\ 3 & 4 & -5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ -1 & 5 \end{bmatrix}$ *the find* $A + B'$.

Solution : $A + B' = \begin{bmatrix} -2 & 1 & 0 \\ 3 & 4 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 4 & -1 \\ 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 5 & -1 \\ 5 & 7 & 0 \end{bmatrix}$.

2. Problem: *If* $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$ *then find* AA'. *Do* A *and* A' *commute with respect to multiplication of matrices?*

Solution : $A' = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$

$$AA' = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} (-1).(-1)+2.2 & (-1).0+2.1 \\ 0.(-1)+1.2 & 0.0+1.1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}.$$

$$A'A = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (-1).(-1)+0.0 & (-1).2+0.1 \\ 2.(-1)+1.0 & 2.2+1.1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}.$$

Since $AA' \neq A'A$, A and A' do not commute.

3. Problem: *If* $A=\begin{bmatrix} 0 & 4 & -2 \\ -4 & 0 & 8 \\ 2 & -8 & x \end{bmatrix}$ *is a skew symmetric matrix, find the value of* x.

Solution : A is a skew symmetric matrix and x is an element of the diagonal. Hence $x = 0$.

4. Problem : *For any* $n\times n$ *matrix* A, *prove that* A *can be uniquely expressed as a sum of a symmetric matrix and a skew symmetric matrix.*

Solution : $A + A'$ is symmetric and $A - A'$ is a skew-symmetric matrix and

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

To prove uniqueness, let B be a symmetric matrix and C be a skew symmetric matrix such that A = B + C.

$$\text{Then } A' = (B+C)' = B' + C' = B - C$$

$$\text{and hence } B = \frac{1}{2}(A + A')$$

$$\text{and } C = \frac{1}{2}(A - A') .$$

Exercise 3(c)

I.

1. If $A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix}$ then find $(AB')'$.

2. If $A = \begin{bmatrix} -2 & 1 \\ 5 & 0 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 & 1 \\ 4 & 0 & 2 \end{bmatrix}$ then find $2A + B'$ and $3B' - A$.

3. If $A = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix}$ then find $A + A'$ and AA'.

4. If $A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & x & 7 \end{bmatrix}$ is a symmetric matrix, then find x.

5. If $A = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -2 \\ -1 & x & 0 \end{bmatrix}$ is a skew symmetric matrix, then find x.

6. Is $\begin{bmatrix} 0 & 1 & 4 \\ -1 & 0 & 7 \\ -4 & -7 & 0 \end{bmatrix}$ symmetric or skew symmetric?

II. **1.** If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, show that $AA' = A'A = I$.

2. If $A = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 4 & 0 \\ 3 & -1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 5 \\ 1 & 2 & 0 \end{bmatrix}$ then find $3A - 4B'$.

3. If $A = \begin{bmatrix} 7 & -2 \\ -1 & 2 \\ 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -1 \\ 4 & 2 \\ -1 & 0 \end{bmatrix}$ then find AB' and BA'.

4. For any square matrix A, show that AA' is symmetric.

3.4 Determinants

Consider the system of two linear equations in two variables,

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

where $c_1 \neq 0$ or $c_2 \neq 0$.

We have learnt in lower classes that this system has a unique solution or not according as $a_1b_2 - a_2b_1$ is not zero or zero. In other words, $a_1b_2 - a_2b_1$ determines whether the system has a unique solution or not and hence it is called the **'determinant'** of the system. Hence we associate the value $a_1b_2 - a_2b_1$ to the matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and call it the determinant (simply determinant) of the matrix.

The determinant of 1×1 matrix is defined as its element.

In this section, we define the determinant of a 3×3 matrix, study its properties and the methods of evaluation of certain determinants.

3.4.1 Definition (Minor of an element)

Consider a square matrix $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

The minor of an element in this matrix is defined as the determinant of the 2×2 *matrix, obtained after deleting the row and the column in which the element is present.*

For example the minor of a_2 is the det. of $\begin{bmatrix} b_1 & c_1 \\ b_3 & c_3 \end{bmatrix} = b_1 c_3 - b_3 c_1$

and the minor of b_3 is the det. of $\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} = a_1 c_2 - a_2 c_1$.

3.4.2 Definition (Cofactor of an element)

The ***cofactor*** *of an element in the* i^{th} *row and the* j^{th} *column of a* 3×3 *matrix is defined as its minor multiplied by* $(-1)^{i+j}$.

We denote the cofactor of a_{ij} by A_{ij}.

For example, consider the matrix in 3.4.1.

Since a_2 is in 2^{nd} row and 1^{st} column, we have

$$A_2 = \text{cofactor of } a_2 = (-1)^{2+1}(b_1 c_3 - b_3 c_1)$$
$$= -(b_1 c_3 - b_3 c_1)$$
$$= b_3 c_1 - b_1 c_3$$

Since b_3 is in 3^{rd} row and 2^{nd} column, we have

$$B_3 = \text{cofactor of } b_3$$
$$= (-1)^{3+2}(a_1 c_2 - a_2 c_1)$$
$$= a_2 c_1 - a_1 c_2.$$

3.4.3 Example

In the matrix $\begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ 4 & 5 & 6 \end{bmatrix}$ *we list out here under, the minors and cofactors of all the elements.*

element a_{ij}	element present in row i, column j	Minor of a_{ij}	Cofactor of a_{ij}
1	1 1	$\begin{vmatrix} -1 & 2 \\ 5 & 6 \end{vmatrix} = -16$	$(-1)^{1+1}(-16) = -16$
0	1 2	$\begin{vmatrix} 3 & 2 \\ 4 & 6 \end{vmatrix} = 10$	$(-1)^{1+2}(10) = -10$
-2	1 3	$\begin{vmatrix} 3 & -1 \\ 4 & 5 \end{vmatrix} = 19$	$(-1)^{1+3}(19) = 19$
3	2 1	$\begin{vmatrix} 0 & -2 \\ 5 & 6 \end{vmatrix} = 10$	$(-1)^{2+1}(10) = -10$
-1	2 2	$\begin{vmatrix} 1 & -2 \\ 4 & 6 \end{vmatrix} = 14$	$(-1)^{2+2}(14) = 14$
2	2 3	$\begin{vmatrix} 1 & 0 \\ 4 & 5 \end{vmatrix} = 5$	$(-1)^{2+3}(5) = -5$
4	3 1	$\begin{vmatrix} 0 & -2 \\ -1 & 2 \end{vmatrix} = -2$	$(-1)^{3+1}(-2) = -2$
5	3 2	$\begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix} = 8$	$(-1)^{3+2}(8) = -8$
6	3 3	$\begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} = -1$	$(-1)^{3+3}(-1) = -1$

3.4.4 Definition (Determinant)

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$. *The sum of the products of elements of the first row with their corresponding cofactors is called the* ***determinant*** *of* A.

The determinant of the matrix $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is written as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

We also denote the determinant of the matrix A by det A or $|A|$.

$$\det A = a_1A_1 + b_1B_1 + c_1C_1$$

So far we have defined the concept of determinant for square matrices of order n for $n = 1, 2, 3$. The concept can be extended to the case $n \geq 4$ also using the principle of mathematical induction. Let $n \geq 4$ and suppose that we know the definition of determinant for square matrices of order $n - 1$. Let $A = [a_{ij}]_{n\times n}$. Then the determinant of A is defined as $\sum_{j=1}^{n} a_{1j} A_{1j}$, where A_{1j} is the cofactor of a_{1j}.

3.4.5 Example

Let us find the determinant of $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ 4 & 5 & 6 \end{bmatrix}$

det A = sum of the products of elements of the first row with their corresponding cofactors

= 1 (cofactor of 1) + 0 (cofactor of 0) + (−2) cofactor of (−2)

= 1(−16) + (−2) (19)

= −16−38 = −54.

3.4.6 Note

The definition of the determinant is formulated by using the elements of the first row and the corresponding cofactors only. However the process can be adopted for the elements of any row or column and the corresponding cofactors. We thus have

$$\det A = \sum_{j=1}^{n} a_{ij} A_{ij} \quad \text{for} \quad 1 \leq i \leq n.$$

Here the sum on the right hand side is independent of i.

If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

then observe that $\det A = a_1A_1 + b_1B_1 + c_1C_1$ expansion along first row

similarly $\det A = a_2A_2 + b_2B_2 + c_2C_2$ expansion along second row

$= a_3A_3 + b_3B_3 + c_3C_3$ expansion along third row

$= a_1A_1 + a_2A_2 + a_3A_3$ expansion along first column

$= b_1B_1 + b_2B_2 + b_3B_3$ expansion along second column

$= c_1C_1 + c_2C_2 + c_3C_3$ expansion along third column

For instance, consider

$$a_1A_1 + a_2A_2 + a_3A_3$$

$$= a_1(-1)^{1+1}\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_2(-1)^{2+1}\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3(-1)^{3+1}\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

$$= a_1A_1 + b_1B_1 + c_1C_1 = \det A .$$

3.4.7 Examples

1. Find the determinant of $A = \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix}$.

Solution: $\det A = 1 \cdot 1 - (-3)(-1) = 1 - 3 = -2$.

2. Find the minors of -1 and 3 in the matrix $\begin{bmatrix} 2 & -1 & 4 \\ 0 & -2 & 5 \\ -3 & 1 & 3 \end{bmatrix}$.

Solution: Minor of $-1 = \begin{vmatrix} 0 & 5 \\ -3 & 3 \end{vmatrix} = 0.3 - (-3).5 = 15 \cdot$

Minor of $3 = \begin{vmatrix} 2 & -1 \\ 0 & -2 \end{vmatrix} = 2.(-2) - 0 \cdot (-1) = -4.$

3. Find the cofactors of the elements 2, -5 in the matrix $\begin{bmatrix} -1 & 0 & 5 \\ 1 & 2 & -2 \\ -4 & -5 & 3 \end{bmatrix}$.

Solution : The element 2 is $(2, 2)^{th}$ element of the given matrix.

$$\text{Hence cofactor of } 2 = (-1)^{2+2}\begin{vmatrix} -1 & 5 \\ -4 & 3 \end{vmatrix}$$

$$= (-1) \cdot 3 - (-4) \cdot 5$$

$$= -3 + 20 = 17 .$$

The element -5 is $(3, 2)^{th}$ element of the given matrix.

$$\text{Hence cofactor of } -5 = (-1)^{3+2}\begin{vmatrix} -1 & 5 \\ 1 & -2 \end{vmatrix}$$

$$= -[(-1) \cdot (-2) - 1 \cdot 5]$$

$$= -(2 - 5) = 3.$$

4. Find the determinant of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 4 \\ -4 & -2 & 5 \end{bmatrix}$.

Solution: Cofactor of 1 $= (-1)^{1+1}\begin{vmatrix} 0 & 4 \\ -2 & 5 \end{vmatrix} = 8$.

Cofactor of -1 $= (-1)^{1+2}\begin{vmatrix} 3 & 4 \\ -4 & 5 \end{vmatrix} = -31$.

Cofactor of 2 $= (-1)^{1+3}\begin{vmatrix} 3 & 0 \\ -4 & -2 \end{vmatrix} = -6$.

$$\text{Now } \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 4 \\ -4 & -2 & 5 \end{vmatrix} = 1\cdot 8 + (-1)\cdot(-31) + 2\cdot(-6) = 8 + 31 - 12 = 27.$$

3.4.8 Properties of determinants

(i) If each element of a row (or column) of a square matrix is zero, then the determinant of that matrix is zero.

The value of the determinant of such a matrix can be easily found to be zero by expanding it along a row (column) containing zeros.

(ii) If two rows (or columns) of a square matrix are interchanged, then the sign of the determinant changes.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

(B is obtained by interchanging first and second rows of A)

$$\det B = a_1(-1)^{2+1}(b_2c_3 - b_3c_2) + b_1(-1)^{2+2}(a_2c_3 - a_3c_2) + c_1(-1)^{2+3}(a_2b_3 - a_3b_2)$$

$$= -[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)]$$

$$= -\det A.$$

(iii) If each element of a row (or column) of a square matrix is multiplied by a number k, then the determinant of the matrix obtained is k times the determinant of the given matrix.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad B = \begin{bmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{bmatrix}$$

(B is obtained by multiplying the elements of first column of A by k)

If the cofactors of a_1, a_2, a_3 in A are A_1, A_2, A_3 then the cofactors of ka_1, ka_2, ka_3 in B are also A_1, A_2, A_3 respectively. Hence

$$\det B = ka_1 A_1 + ka_2 A_2 + ka_3 A_3$$
$$= k(a_1 A_1 + a_2 A_2 + a_3 A_3)$$
$$= k(\det A).$$

(iv) If A is square matrix of order 3 and k is a scalar, then $|kA| = k^3|A|$. By applying property (iii), three times, we get the result.

(v) If two rows (or columns) of a square matrix are identical, then the determinant of that matrix is zero.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{bmatrix}$$

(second and third rows are identical)

$$\text{Then } \det A = a_1 A_1 + b_1 B_1 + c_1 C_1$$
$$= a_1(0) + b_1(0) + c_1(0) = 0.$$

(vi) If the corresponding elements of two rows (or columns) of a square matrix are in the same ratio, then the determinant of that matrix is zero.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Then

$$\det A = \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{by property (iii)}$$

$$= k(0) \quad \text{by property (v)}$$
$$= 0.$$

(vii) If each element in a row (or column) of a square matrix is the sum of two numbers, then its determinant can be expressed as the sum of the determinants of two square matrices as shown below.

$$\text{Let } A = \begin{bmatrix} a_1 + x_1 & b_1 & c_1 \\ a_2 + x_2 & b_2 & c_2 \\ a_3 + x_3 & b_3 & c_3 \end{bmatrix}, B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, C = \begin{bmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{bmatrix}$$

If in A, the cofactors of $a_1+x_1,\ a_2+x_2, a_3+x_3$ are A_1, A_2, A_3 then the cofactors of $a_1,\ a_2, a_3$ in B and of x_1, x_2, x_3 in C are also A_1, A_2, A_3 respectively.

Now,

$$\begin{aligned}\det A &= (a_1+x_1)A_1+(a_2+x_2)\ A_2+(a_3+x_3)A_3\\ &= (a_1A_1+a_2A_2+a_3A_3)+(x_1A_1+x_2A_2+x_3A_3)\\ &= \det B+\det C.\end{aligned}$$

$$\therefore\quad \begin{vmatrix} a_1+x_1 & b_1 & c_1\\ a_2+x_2 & b_2 & c_2\\ a_3+x_3 & b_3 & c_3\end{vmatrix}=\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}+\begin{vmatrix} x_1 & b_1 & c_1\\ x_2 & b_2 & c_2\\ x_3 & b_3 & c_3\end{vmatrix}.$$

(viii) If each element of a row (or column) of a square matrix is multiplied by a number k and added to the corresponding element of another row (or column) of the matrix, then the determinant of the resultant matrix is equal to the determinant of the given matrix.

$$\text{Let } A=\begin{bmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{bmatrix} \text{ and } B=\begin{bmatrix} a_1 & b_1 & c_1\\ a_2+ka_1 & b_2+kb_1 & c_2+kc_1\\ a_3 & b_3 & c_3\end{bmatrix}$$

(B is obtained from A by multiplying each element of the 1^{st} row of A by k and then adding them to the corresponding elements of the 2^{nd} row of A)

$$\det B=\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}+\begin{vmatrix} a_1 & b_1 & c_1\\ ka_1 & kb_1 & kc_1\\ a_3 & b_3 & c_3\end{vmatrix}\quad \text{by property (vii)}$$

$$=\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}+0\quad \text{by property (vi)}$$

$$=\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}=\det A.$$

(ix) The sum of the products of the elements of a row (or column) with the cofactors of the corresponding elements of another row (or column) of a square matrix is zero.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

Consider the sum of the products of the elements of the second row with the cofactors of the corresponding elements of the first row.,

$$\text{i.e., } a_2 A_1 + b_2 B_1 + c_2 C_1$$

$$= a_2 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_2 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_2 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \quad \text{by property (v).}$$

(x) If the elements of a square matrix are polynomials in x and its determinant is zero when $x = a$, then $x - a$ is a factor of the determinant of the matrix.

$$\text{Let } A(x) = \begin{bmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{bmatrix}.$$

Now det [A(x)] is a polynomial in x.

If det $[A(a)] = 0$ then by Remainder theorem, $x - a$ is a factor of det $[A(x)]$.

(xi) For any square matrix A, det A = det (A$'$).

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \text{ then } A' = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

The values of the cofactors of a_1, b_1, c_1, are same in both A and A$'$.

Hence det A $= a_1 A_1 + b_1 B_1 + c_1 C_1$ = det A$'$.

(xii) Det (AB) = (det A) (det B) for matirces A, B of order 2.

Consider the matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$,

$$\det A = a_{11} a_{22} - a_{21} a_{12}; \ \det B = b_{11} b_{22} - b_{21} b_{12}.$$

$$\text{Now } AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} b_{11} + a_{12} b_{21} & a_{11} b_{12} + a_{12} b_{22} \\ a_{21} b_{11} + a_{22} b_{21} & a_{21} b_{12} + a_{22} b_{22} \end{bmatrix}$$

$$\det(AB) = (a_{11} b_{11} + a_{12} b_{21})(a_{21} b_{12} + a_{22} b_{22}) - (a_{21} b_{11} + a_{22} b_{21})(a_{11} b_{12} + a_{12} b_{22})$$

$$= a_{11} a_{21} b_{11} b_{12} + a_{11} a_{22} b_{11} b_{22} + a_{12} a_{21} b_{12} b_{21} + a_{12} a_{22} b_{21} b_{22}$$

$$- a_{11} a_{21} b_{11} b_{12} - a_{12} a_{21} b_{11} b_{22} - a_{11} a_{22} b_{12} b_{21} - a_{12} a_{22} b_{21} b_{22}$$

$$= a_{11} a_{22} b_{11} b_{22} + a_{12} a_{21} b_{12} b_{21} - a_{12} a_{21} b_{11} b_{22} - a_{11} a_{22} b_{12} b_{21}$$

$$= a_{11} a_{22} (b_{11} b_{22} - b_{12} b_{21}) - a_{12} a_{21} (b_{11} b_{22} - b_{12} b_{21})$$

$$= (a_{11} a_{22} - a_{12} a_{21})(b_{11} b_{22} - b_{12} b_{21})$$

$$= (\det A)(\det B).$$

If A and B are matrices of order three then also in a similar manner we can show that

$$\det(AB) = (\det A)(\det B).$$

This is true in general, for all matrices of order *n;* the proof of this is beyond the scope of this book.

(xiii) For any positive integer *n,* $\det(A^n) = (\det A)^n$.

(xiv) If A is a triangular matrix (upper or lower), then determinant of A is the product of the diagonal elements.

3.4.9 Notation

While evaluating determinants, we use the following notations.

(i) $R_1 \leftrightarrow R_2$, to mean that the rows R_1 and R_2 are interchanged.

(ii) $R_1 \to kR_1$, to mean that the elements of R_1 are multiplied by *k.*

(iii) $R_1 \to R_1 + kR_2$ to mean that the elements of R_1 are added with *k* times the corresponding elements of R_2.

Similar notation is used for other rows and columns.

3.4.10 Solved Problems

1. Problem : *Show that* $\begin{vmatrix} 1 & a & a^2 \\ 1 & a & b^2 \\ 1 & a & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$.

Solution : L.H.S. $= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

On applying $R_2 \to (R_2 - R_1)$; $R_3 \to (R_3 - R_1)$ on LHS we get

$$\text{L.H.S.} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

On expanding the det. along the first column, we get

$$= 1.\begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$$

$$= (a-b)(b-c)(c-a) = \text{R.H.S}.$$

2. Problem : *Without expanding the determinant show that*

$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

Solution :

$$\text{L.H.S.} = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \text{ (by applying } R_1 \to R_1 + R_2 + R_3\text{)}$$

$$= 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix} \text{ (by applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1\text{)}$$

$$= 2\begin{vmatrix} a & b & c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix} \qquad \text{(by applying } R_1 \to R_1 + R_2 + R_3\text{)}$$

$$= 2(-1)(-1)\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \text{R.H.S.}$$

3. Problem : *Show that* $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$.

Solution :

$$\text{L.H.S.} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} 0 & a^2-c^2 & a^3-c^3 \\ 0 & b^2-c^2 & b^3-c^3 \\ 1 & c^2 & c^3 \end{vmatrix} \quad \text{(by applying } R_1 \to R_1 - R_3;\ R_2 \to R_2 - R_3\text{)}$$

$$= (a-c)(b-c)\begin{vmatrix} 0 & a+c & a^2+ac+c^2 \\ 0 & b+c & b^2+bc+c^2 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= (a-c)(b-c)\begin{vmatrix} 0 & a+c & a^2+ac+c^2 \\ 0 & b-a & b^2-a^2+bc-ac \\ 1 & c^2 & c^3 \end{vmatrix} \quad \text{(by applying } R_2 \to R_2 - R_1\text{)}$$

$$= (a-c)(b-c)(b-a)\begin{vmatrix} 0 & a+c & a^2+ac+c^2 \\ 0 & 1 & c+a+b \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= (a-c)(b-c)(b-a)\begin{vmatrix} a+c & a^2+ac+c^2 \\ 1 & a+b+c \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)(ab+bc+ca) = \text{R.H.S.}$$

4. Problem : *If ω is complex (non real) cube root of 1 then show that* $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0$.

Solution : Method 1

$$\text{L.H.S.} = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$

$$= \begin{vmatrix} 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} \text{ by applying } R_1 \to R_1 + R_2 + R_3$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0 = \text{R.H.S.} \ (\because 1+\omega+\omega^2 = 0).$$

Method 2

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = \begin{vmatrix} \omega^3 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = \omega \begin{vmatrix} \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0.$$

5. Problem: *Show that* $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$.

Solution: $\text{L.H.S.} = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \text{ applying } R_1 \to R_1 + R_2 + R_3$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix} \quad \text{applying } C_2 \to C_2 - C_1, \; C_3 \to C_3 - C_1$$

$$= (a+b+c)\begin{vmatrix} -b-c-a & 0 \\ 0 & -c-a-b \end{vmatrix} = (a+b+c)^3 = \text{R.H.S.}$$

6. Problem: *Show that the determinant of skew- symmetric matrix of order three is always zero.*

Solution : Let us consider a skew-symmetrc matrix of order three, say -

$$A = \begin{bmatrix} 0 & -c & -b \\ c & 0 & -a \\ b & a & 0 \end{bmatrix} \quad \Rightarrow \quad |A| = \begin{vmatrix} 0 & -c & -b \\ c & 0 & -a \\ b & a & 0 \end{vmatrix}$$

$$= +c(0 + ab) - b(ca - 0) = abc - abc = 0.$$

Hence $|A| = 0$.

Observe that the determinant of skew-symmetric matrix of order two need not be zero.

For example $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ is skew symmetric matrix of order 2, and det $A \neq 0$.

7. Problem : *Find the value of x if* $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$.

Solution : $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix}$

$$= \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ -2 & -6 & -12 \\ -6 & -24 & -60 \end{vmatrix} \text{applying } R_2 \to (R_2 - R_1),\; R_3 \to (R_3 - R_1)$$

$$= (-2)(-6)\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix}$$

Now given that $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} = 0$.

Expanding the determinant along the elements of first row, we have

$$(x-2)(30-24)-(2x-3)(10-6)+(3x-4)(4-3)=0$$

i.e., $6(x-2)-4(2x-3)+(3x-4)=0$

i.e., $x-4=0$. Hence $x=4$.

Exercise 3(d)

I. **1.** Find the determinants of the following matrices.

(i) $\begin{bmatrix} 2 & 1 \\ 1 & -5 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 4 & 5 \\ -6 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ **(iv)** $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & -1 & 4 \\ -3 & 7 & 6 \end{bmatrix}$ **(vi)** $\begin{bmatrix} 2 & -1 & 4 \\ 4 & -3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

(vii) $\begin{bmatrix} 1 & 2 & -3 \\ 4 & -1 & 7 \\ 2 & 4 & -6 \end{bmatrix}$ **(viii)** $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

(ix) $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ **(x)** $\begin{bmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{bmatrix}$

2. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 5 & -6 & x \end{bmatrix}$ and $\det A = 45$ then find x.

II. **1.** Show that $\begin{vmatrix} bc & b+c & 1 \\ ca & c+a & 1 \\ ab & a+b & 1 \end{vmatrix} = (a-b)(b-c)(c-a)$.

2. Show that $\begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ a & b & c \end{vmatrix} = a^3+b^3+c^3-3abc$.

3. Show that $\begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix} = 4xyz$.

4. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and $\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} \neq 0$ then show that $abc = -1$.

5. Without expanding the determinant, prove that (i) $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$.

(ii) $\begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ yz & zx & xy \end{vmatrix}$ (iii) $\begin{vmatrix} 1 & bc & b+c \\ 1 & ca & c+a \\ 1 & ab & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$.

6. If $\Delta_1 = \begin{vmatrix} a_1^2+b_1+c_1 & a_1a_2+b_2+c_2 & a_1a_3+b_3+c_3 \\ b_1b_2+c_1 & b_2^2+c_2 & b_2b_3+c_3 \\ c_3c_1 & c_3c_2 & c_3^2 \end{vmatrix}$ and

$\Delta_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then find the value of $\frac{\Delta_1}{\Delta_2}$.

7. If $\Delta_1 = \begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}$ and $\Delta_1 = \Delta_2$, then show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

III. 1. Show that $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$.

2. Show that

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = (a^3+b^3+c^3-3abc)^2.$$

3. Show that $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$.

4. Show that $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$.

5. Show that $\begin{vmatrix} -2a & a+b & c+a \\ a+b & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(a+b)(b+c)(c+a)$.

6. Show that $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$.

7. Show that $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0$.

8. Show that $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x+2a)(x-a)^2$.

3.5 Adjoint and Inverse of a Matrix

In this section, we define the concepts of invertibility of a matrix and the multiplicative inverse of an invertible matrix and study certain properties of inverses and provide a method of finding the multiplicative inverse of a given invertible matrix.

3.5.1 Definition (Singular and Non-singular matrices)

A square matrix is said to be ***singular*** *if its determinant is zero. Otherwise it is said to be* ***non-singular.***

For example, $\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$ is a singular matrix while $\begin{bmatrix} 3 & -2 \\ 6 & 4 \end{bmatrix}$ is non-singular.

3.5.2 Definition (Adjoint of a matrix)

The transpose of the matrix formed by replacing the elements of a square matrix A (of order greater than one) with the corresponding cofactors is called the Adjoint of A and is denoted by Adj A.

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ and A_i, B_i, C_i be the cofactors of a_i, b_i, c_i respectively.

Then $\text{Adj } A = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}^T = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}.$

3.5.3 Definition (Invertible matrix)

Let A be a square matrix. We say that A is invertible if a matrix B exists such that AB = BA = I, where I is the unit matrix of the same order as A and B.

3.5.4 Note

(i) For the products AB and BA to be both defined and equal, it is necessary that A and B are both square matrices of the same order. Thus, non-square matrices are not invertible.

(ii) If A is invertible, then A is non-singular, hence det A $\neq$ 0.

[Let A be invertible. Then there exsists a matrix B such that AB = I.
Hence (det A) (det B) = det (AB) = det I = 1. Hence det A $\neq$ 0].

(iii) If B exists; such that AB = BA = I, then such a B is unique and is denoted by A^{-1} and is called the **multiplicative inverse** or inverse of A.

[For, if B and C are inverses of A, then by definition AB = BA = I and AC = CA = I. Then B = BI = B (AC) = (BA)C = IC = C].

3.5.5 Theorem

Let A and B be invertible matrices. Then A^{-1}, A' and AB are invertible. Further

(i) $(A^{-1})^{-1} = A$

(ii) $(A')^{-1} = (A^{-1})'$

(iii) $(AB)^{-1} = B^{-1}A^{-1}$.

Proof: (i) Let $A^{-1} = C$. Then CA = CA = I.

By 3.5.4 (ii), C is invertible and the multiplicative inverse of C is A

i.e., $C^{-1} = A$

i.e., $(A^{-1})^{-1} = A$.

(ii) Consider

$$(A')(A^{-1})' = (A^{-1}A)' = I' = I. \qquad \text{... (1)}$$

Similarly,

$$(A^{-1})'(A') = (AA^{-1})' = I' = I. \qquad \text{... (2)}$$

From (1) and (2)

$$(A')(A^{-1})' = (A^{-1})'(A') = I.$$

$\therefore$ By definition A' is invertible and the multiplicative inverse of A' is $(A^{-1})'$.

i.e., $(A')^{-1} = (A^{-1})'$.

(iii) Since A and B are invertible, we have

$$AA^{-1} = A^{-1}A = I \quad \ldots (1)$$

and $$BB^{-1} = B^{-1}B = I. \quad \ldots (2)$$

Now

$$(AB)\,(B^{-1}A^{-1})$$

$$= A\,(BB^{-1})\,A^{-1}, \text{ by Associative law}$$

$$= A\,(I)\,A^{-1}, \text{ by (2)}$$

$$= (A\,I)\,A^{-1}$$

$$= AA^{-1} = I, \text{ by (1)}.$$

Similarly

$$(B^{-1}\,A^{-1})\,(AB) = B^{-1}\,(A^{-1}\,A)\,B \;= B^{-1}\,(I)\,B$$

$$= B^{-1}\,(IB) \;= B^{-1}B = I.$$

$\therefore$ We have $(AB)(B^{-1}\,A^{-1}) = (B^{-1}\,A^{-1})\,(AB) = I$.

Hence by definition AB is invertible and the multiplicative inverse of AB is $B^{-1}\,A^{-1}$.

i.e., $(AB)^{-1} = B^{-1}\,A^{-1}$.

3.5.6 Theorem

If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is a non-singular matrix then A is invertible and $A^{-1} = \dfrac{\text{Adj}\,A}{\det A}$.

Proof : By definition,

$$\text{Adj } A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}.$$

$$\text{Now } A\cdot(\text{Adj } A) = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1A_1+b_1B_1+c_1C_1 & a_1A_2+b_1B_2+c_1C_2 & a_1A_3+b_1B_3+c_1C_3 \\ a_2A_1+b_2B_1+c_2C_1 & a_2A_2+b_2B_2+c_2C_2 & a_2A_3+b_2B_3+c_2C_3 \\ a_3A_1+b_3B_1+c_3C_1 & a_3A_2+b_3B_2+c_3C_2 & a_3A_3+b_3B_3+c_3C_3 \end{bmatrix}$$

$$= \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix}$$

$$= \det A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= (\det A)\,I.$$

Now, since $\det A \neq 0$, we have $A.\dfrac{\text{Adj}\,A}{\det A} = I$.

Similarly, we can show that $\dfrac{\text{Adj}\,A}{\det A}.A = I$.

Let $B = \dfrac{\text{Adj}\,A}{\det A}$. Then $AB = BA = I$.

Hence A is invertible and $A^{-1} = B = \dfrac{\text{Adj}\,A}{\det A}$.

3.5.7 Solved Problems

1. Problem: *Find the adjoint and the inverse of the matrix* $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$.

Solution $\det A = \begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} = 1.(-5) - 3.2 = -5-6 = -11 \neq 0.$

Hence A is invertible.

The cofactor matrix of $A = \begin{bmatrix} -5 & -3 \\ -2 & 1 \end{bmatrix}$

$$\therefore \text{Adj}\ A = \begin{bmatrix} -5 & -3 \\ -2 & 1 \end{bmatrix}' = \begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}.$$

$$\therefore A^{-1} = \frac{\text{Adj}\,A}{\det A} = \frac{1}{-11}\begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{11} & \frac{2}{11} \\ \frac{3}{11} & \frac{-1}{11} \end{bmatrix}.$$

2. Problem : *Find the adjoint and the inverse of the matrix* $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$.

Solution: $\det A = 1\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix}$

$$= 1(16-9) - 3(4-3) + 3(3-4) = 7-3-3 = 1 \neq 0.$$

Therefore A is invertible.

The cofactor matrix of A is $B = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

$$\therefore \text{Adj}\ A = B' = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{Adj}\,A}{\det A} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad (\because \det A = 1).$$

3. Problem: *Show that* $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$ *is non - singular and find* A^{-1}.

Solution : $\det A = 1\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 3 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1-6+1 = -4 \neq 0.$

Hence A is a non-singular matrix.

The cofactor matrix of A is B $= \begin{bmatrix} 1 & -3 & 1 \\ -3 & 1 & 1 \\ 4 & 0 & -4 \end{bmatrix}$

The transpose of B is the adjoint of A.

$$\therefore \text{ Adj A} = B' = \begin{bmatrix} 1 & -3 & 4 \\ -3 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{Adj A}}{\det A} = \frac{1}{-4}\begin{bmatrix} 1 & -3 & 4 \\ -3 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & -1 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix}.$$

Exercise 3(e)

I **1.** Find the adjoint and the inverse of the following matrices.

(i) $\begin{bmatrix} 2 & -3 \\ 4 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}$

2. If A $= \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$, $a^2+b^2+c^2+d^2=1$ then find the inverse of A.

3. If A $= \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$, then find $(A')^{-1}$.

4. If $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$, then show that the adjoint of A is $3A'$. Find A^{-1}.

5. If $abc \neq 0$, find the inverse of $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$.

II 1. If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $B = \frac{1}{2}\begin{bmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{bmatrix}$, then show that ABA^{-1} is a diagonal matrix.

2. If $3A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$, then show that $A^{-1} = A'$.

3. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, then show that $A^{-1} = A^3$.

4. If $AB = I$ or $BA = I$, then prove that A is invertible and $B = A^{-1}$.

3.6 Consistency and Inconsistency of system of Simultaneous Equations - Rank of a Matrix

We devote this section for the study of the rank of a matrix, existence and the nature of solutions of a system of linear equations - homogeneous and non-homogeneous, in two and three variables.

Consider the following system of simultaneous non-homogeneous linear equations (two equations in two variables):

$$\left.\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}\right\} \quad \ldots.. \text{ I}$$

These equations can be represented as a matrix equation as A X = D, where

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$ is called the coefficient matrix.

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$ is called the variable matrix,

and $$D = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$ is called the constant matrix.

AX = D is the matrix representation of the equations given in system I, for

$$AX = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 x + b_1 y \\ a_2 x + b_2 y \end{bmatrix}$$

AX = D becomes $$\begin{bmatrix} a_1 x + b_1 y \\ a_2 x + b_2 y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

and corresponding elements of two equal matrices are equal.

The coefficient matrix augmented with the constant column matrix, is called the **Augmented matrix**, generally denoted by [A D]. Hence, the augmented matrix of system I is

$$[A\ D] = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}.$$

We listed the various systems of equations, along with the corresponding matrix equations and matrices involved in the following tabular form.

Types of systems of simultaneous linear equations

Nature of the system	Equations	Matrix equation	Corresponding Matrices: Coefficient matrix A	Corresponding Matrices: Variables matrix X	Corresponding Matrices: Constant matrix D	Augmented matrix [AD]
I. Non-homogeneous pair of equations in two unknowns	$a_1x+b_1y=c_1$ $a_2x+b_2y=c_2$ $c_1\neq 0$ or $c_2\neq 0$	AX = D	$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$	$\begin{bmatrix} x \\ y \end{bmatrix}$	$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$	$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$
II. Non-homogeneous triad of equations in three unknowns	$a_1x+b_1y+c_1z=d_1$ $a_2x+b_2y+c_2z=d_2$ $a_3x+b_3y+c_3z=d_3$ $d_1\neq 0$ or $d_2\neq 0$ or $d_3\neq 0$	AX = D	$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$	$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$	$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$	$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$
III. Pair of homogeneous equations in two unknowns	$a_1x+b_1y=0$ $a_2x+b_2y=0$	AX = O	$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$	$\begin{bmatrix} x \\ y \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \end{bmatrix}$
IV. Three homogeneous equations in three unknowns	$a_1x+b_1y+c_1z=0$ $a_2x+b_2y+c_2z=0$ $a_3x+b_3y+c_3z=0$	AX = O	$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$	$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ a_3 & b_3 & c_3 & 0 \end{bmatrix}$

Here we confine to the above types of systems of equations in three variables. Before solving the systems of equations, we first study an important concept namely the rank of a matrix.

3.6.1 Definition (Submatrix)

A matrix obtained by deleting some rows or columns (or both) of a matrix is called a ***submatrix*** *of the given matrix.*

For example, If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -1 & 2 & 0 \end{bmatrix}$.

Then some submatrices of A are

$\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$ – obtained by deleting R_2 and C_3 of A

$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ – obtained by deleting R_3 of A

$\begin{bmatrix} 2 & 3 \\ 3 & 1 \\ 2 & 0 \end{bmatrix}$ – obtained by deleting C_1 of A

[0] – obtained by deleting R_1, R_2, C_1 and C_2 of A.

3.6.2 Definition (Rank of a matrix)

Let A be a non-zero matrix. The ***rank*** *of A is defined as the maximum of the orders of the non-singular square submatrices of A. The rank of a null matrix is defined as zero. The rank of a matrix A is denoted by* ***rank (A)****.*

3.6.3 Note

If A is a non-zero matrix of order 3, then the rank of A is

(i) 1 if every 2×2 submatrix is singular

(ii) 2 if A is singular and atleast one of its 2×2 submatrices is non-singular

(iii) 3 if A is non-singular.

3.6.4 Examples

1. $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

det A = -5. A is non-singular, and hence rank (A) = 3.

2. $B = \begin{bmatrix} -1 & -2 & -3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

det B = 0, Hence rank (B) $\neq$ 3.

Now $\begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}$ is a submatrix of B, whose determinant is 2.

Hence rank (B) = 2.

3. $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

C is a matrix of order 3×4.

Let $C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Then C_1 is a square submatrix of C of order 3 and det $C_1 = 1$.

Hence rank of the given matrix is 3.

3.6.5 Definition (Elementary Transformations)

The following tranformations are known as ***elementary transformations*** *on a matrix.*

(i) *Interchange of two rows (or columns).*

(ii) *Multiplication of elements of a row (or column) by a non-zero number*

(iii) *Addition to the elements of a row (or a column), the corresponding elements of another row (or column) multiplied by any non-zero number.*

Elementary transformations enable us to transform a given matrix into triangular matrix. In a triangular matrix, the search for the highest order non-singular submatrices is easier. (Why!) We state here below a theorem without proof, which enables us to determine the rank of a matrix using elementary transformations.

3.6.6 Theorem

Elementary transformations on a matrix do not change its rank.

A matrix obtained from a given matrix by applying a finite number of elementary transformations (in succession) is said to be equivalent to it. If A and B are equivalent, we write $A \sim B$.

3.6.7 Solved Problems

1. Problem: *Find the rank of* $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ *using elementary transformations.*

Solution: $A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ (on interchanging R_1 and R_2)

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -4 & -8 \end{bmatrix} \text{(on applying } R_3 \to R_3 - 3R_1)$$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{(on applying } R_1 \to R_1 - 2R_2, R_3 \to R_3 + 4R_2)$$

The last matrix is singular and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a non-singular submatrix of it. Hence its rank is 2. Rank (A) = 2.

2.Problem: *Find the rank of* $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$ *using elementary transformations.*

Solution : $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$

$$R_2 \to R_2 - 3R_1 : \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -2 & 1 & 5 \\ -2 & 3 & 2 & 5 \end{bmatrix}$$

$$R_3 \to R_3 + 2R_1 : \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -2 & 1 & 5 \\ 0 & 7 & 2 & 3 \end{bmatrix}$$

$$R_3 \to 2R_3 + 7R_2 : \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -2 & 1 & 5 \\ 0 & 0 & 11 & 41 \end{bmatrix}$$

$$\det \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 11 \end{bmatrix} = -22 \neq 0.$$

$\therefore$ R(A) = 3.

Exercise 3(f)

Find the rank of each of the following matrices.

I. **1.** $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ **5.** $\begin{bmatrix} 1 & 0 & -4 \\ 2 & -1 & 3 \end{bmatrix}$ **6.** $\begin{bmatrix} 1 & 2 & 6 \\ 2 & 4 & 3 \end{bmatrix}$

II. **1.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & 4 & -1 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **5.** $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$ **6.** $\begin{bmatrix} 0 & 1 & 1 & -2 \\ 4 & 0 & 2 & 5 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

3.6.8 Definition (Consistent and Inconsistent systems)

We say that a system of linear equations is

(i) ***consistent*** *if it has a solution.*

(ii) ***inconsistent*** *if it has no solution.*

3.6.9 Solutions of nonhomogeneous system of equations

We consider solving the following system of 3 equations in 3 unknowns

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

This system can be represented by a matrix equation AX = D where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}_{3\times 3}$$ is the coefficient matrix,

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1}$$ is the variable matrix,

$$D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$ is the constant matrix,

$$[A\,D] = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$$ is the augmented matrix.

We state here a theorem without proof, which indicates the nature of solutions of the system.

3.6.10 Theorem

The system of three equations in three unknowns $AX = D$ *has*

(i) *a unique solution if rank* (A) = *rank* ([A D]) = 3.

(ii) *infinitely many solutions if rank* (A) = *rank* ([A D]) < 3.

(iii) *no solution if rank* (A) $\neq$ *rank* ([A D]).

Note that the system is consistent if and only if rank (A) = rank ([A D]).

The method of solving the equations is illustrated in the following example.

3.6.11 Example

Show that the system of equations given below is not consistent.

$$2x + 6y = -11$$
$$6x + 20y - 6z = -3$$
$$6y - 18z = -1$$

Solution: The given system of equations can be written in the form

AX = D, where

$$A = \begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad D = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix}.$$

Consider the augmented matrix

$$[A\,D] = \begin{bmatrix} 2 & 6 & 0 & -11 \\ 6 & 20 & -6 & -3 \\ 0 & 6 & -18 & -1 \end{bmatrix}.$$

On applying $R_2 \to R_2 - 3R_1$, we get

$$[A\,D] \sim \begin{bmatrix} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 6 & -18 & -1 \end{bmatrix}.$$

On applying $R_3 \to R_3 - 3R_2$, we get

$$[A\,D] \sim \begin{bmatrix} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 0 & 0 & -91 \end{bmatrix}.$$

Now rank of [A D] = 3, since the 3 × 3 submatrix

$$\begin{bmatrix} 6 & 0 & -11 \\ 2 & -6 & 30 \\ 0 & 0 & -91 \end{bmatrix},$$

is non - singular (its determinant is $-91\,(6)\,(-6) \neq 0$)

But the rank of the coefficient matrix is not 3 because

$$\det \begin{bmatrix} 2 & 6 & 0 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore$ rank of (A) $\neq$ rank $([A\,D])$.

Hence the given system is inconsistent.

3.6.12 Why do we use only elementary row transformations?

Let us apply elementary column transformations to the augmented matrix of example 3.6.11.

$$[A\,D] \sim \begin{bmatrix} 2 & 6 & 0 & -11 \\ 6 & 20 & -6 & -3 \\ 0 & 6 & -18 & -1 \end{bmatrix}.$$

On applying $C_2 \to C_2 - 3C_1$, we get

$$[A\,D] \sim \begin{bmatrix} 2 & 0 & 0 & -11 \\ 6 & 2 & -6 & -3 \\ 0 & 6 & -18 & -1 \end{bmatrix}.$$

On applying $C_1 \to C_1 - 3C_2$, and $C_3 \to C_3 + 3C_2$, we get

$$[A\,D] \sim \begin{bmatrix} 2 & 0 & 0 & -11 \\ 0 & 2 & 0 & -3 \\ -18 & 6 & 0 & -1 \end{bmatrix}.$$

Now we can easily observe that the rank of the coefficient matrix is $\neq 3$, as

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ -18 & 6 & 0 \end{bmatrix} \text{ is singular.}$$

The rank of the augmented matrix is 3, since the sub matrix

$$\begin{bmatrix} 2 & 0 & -11 \\ 0 & 2 & -3 \\ -18 & 6 & -1 \end{bmatrix} \text{ is non-singular.}$$

(determinant $2(-2+18)-11(36) = 32-11\times 36 \neq 0)$.

Hence rank (A) $\neq$ rank $([A\,D])$. Hence the system is inconsistent.

Thus, we can use either row transformations or column transformations to find whether a system is consistent or inconsistent.

But, if we require the solution of the system also, then elementary row transformation only are useful, as seen by the following discussion.

Each row of the augmented matrix corresponds to an equation of the system.

In the example 3.6.11.

1st row : 2 6 0 –11 corresponds to the first equation $2x+6y=-11$.

2nd row : 6 20 – 6 – 3 corresponds to the second equation $6x+20y-6z=-3$.

3rd row : 0 6 –18 – 1 corresponds to the third equation $6y-18z=-1$.

The following are the effects of elementary row transformations on the equations.

Sl. No.	Elementary row operation	Effect on the equations
1.	Inter change of two rows say R_1 and R_2.	The first equation is numbered as 2 and the second equation is numbered as 1.
2.	Multiplying the elements of the *i-th* row with a non-zero number *k*.	*i-th* equation is multiplied by *k*.
3.	The elements of the *i-th* row are added with *k* times corresponding elements of the *j-th* row $(i \neq j)$.	the *j-th* equation is multiplied with *k* and added to the *i-th* equation.

The effect of the elementary row transformation on the equations is nothing but the steps that we employ for solving the equations under traditional elimination process. As such, we can, at any stage of the problem, write an equivalent system of equations from the augmented matrix. But if we use elementary column transformations we may not obtain an equivalent set of equations.

Hence, if we use Elementary row transformations, we can

(i) decide whether the system is consistent or not and also

(ii) write the solution of the system, if it is consistent.

This is illustrated in the following solved problems.

3.6.13 Solved Problems

1. Problem: *Apply the test of rank to examine whether the following equations are consistent.*

$$2x - y + 3z = 8$$
$$-x + 2y + z = 4$$
$$3x + y - 4z = 0;$$

and if consistent, find the complete solution.

Solution : The augmented matrix is

$$[AD] = \begin{bmatrix} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 2 & 1 & 4 \\ 2 & -1 & 3 & 8 \\ 3 & 1 & -4 & 0 \end{bmatrix} \text{ (on interchanging } R_1 \text{ and } R_2)$$

we transform the above matrix into an upper triangular matrix.

$$\sim \begin{bmatrix} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 7 & -1 & 12 \end{bmatrix} \text{ (on applying } R_2 \to R_2 + 2R_1, R_3 \to R_3 + 3R_1)$$

$$\sim \begin{bmatrix} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 0 & -38 & -76 \end{bmatrix} \text{ (on applying } R_3 \to 3R_3 - 7R_2) \qquad \text{... (F)}$$

$$\text{Now } \det \begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & -38 \end{bmatrix} = (-1)(3)(-38) = 114.$$

Hence rank (A) = rank $([AD])$ = 3.

$\therefore$ By the Theorem of 3.6.10, The system has a unique solution.

We write the equivalent system of equations from (F), i.e.,

$$-x + 2y + z = 4$$
$$3y + 5z = 16$$
$$-38z = -76$$

$\therefore$ $z = 2, y = 2, x = 2$ is the solution.

2. Problem: *Show that the following system of equations is consistent and solve it completely:*

$$x+y+z=3$$
$$2x+2y-z=3$$
$$x+y-z=1$$

Solution : The given equations are equivalent to the equation AX = D, where

$$A=\begin{bmatrix}1&1&1\\2&2&1\\1&1&-1\end{bmatrix},\ X=\begin{bmatrix}x\\y\\z\end{bmatrix} \text{ and } D=\begin{bmatrix}3\\3\\1\end{bmatrix}.$$

Augmented matrix $[A\,D] = \begin{bmatrix}1&1&1&3\\2&2&-1&3\\1&1&-1&1\end{bmatrix}$.

On applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$ we get

$$\sim\begin{bmatrix}1&1&1&3\\0&0&-3&-3\\0&0&-2&-2\end{bmatrix}.$$

On applying $R_3 \to 3R_3 - 2R_2$ we get

$$\sim\begin{bmatrix}1&1&1&3\\0&0&-3&-3\\0&0&0&0\end{bmatrix} \quad \ldots \quad (F)$$

Clearly all the submatrices of order 3 of the above matrix are singular.

Hence rank $(A) \neq 3$, and rank $([A\,D]) \neq 3$.

Now the non-singular matrix $\begin{bmatrix}1&1\\0&-3\end{bmatrix}$ is a submatrix of both A and [A D]. Hence rank (A) = rank $([A\,D]) = 2$.

Hence by Theorem 3.6.10(ii), the system is consistent and has infinitely many solutions.

We now write the equivalent set of equations from (F).

$$x+y+z=3$$
$$-3z=-3$$

Hence $z=1, x+y=2$.

Hence $x=k,\ y=2-k,\ z=1,\ k\in \mathbf{R}$ is the solution set.

Exercise 3 (g)

I Examine whether the following systems of equations are consistent or inconsistent and if consistent find the complete solutions.

1. $x+y+z=4$
   $2x+5y-2z=3$
   $x+7y-7z=5$

2. $x+y+z=6$
   $x-y+z=2$
   $2x-y+3z=9$

3. $x+y+z=1$
   $2x+y+z=2$
   $x+2y+2z=1$

4. $x+y+z=9$
   $2x+5y+7z=52$
   $2x+y-z=0$

5. $x+y+z=6$
   $x+2y+3z=10$
   $x+2y+4z=1$

6. $x-3y-8z=-10$
   $3x+y-4z=0$
   $2x+5y+6z=13$

7. $2x+3y+z=9$
   $x+2y+3z=6$
   $3x+y+2z=8$

8. $x+y+4z=6$
   $3x+2y-2z=9$
   $5x+y+2z=13$

3.7 Solution of Simultaneous Linear Equations

In this section we discuss some methods of solving systems of simultaneous linear equations.

3.7.1. Cramer's Rule

Consider the system of equations

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is non-singular.

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the solution of the equation $AX = D$, where $D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Then $x\Delta = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}$

On applying $C_1 \to C_1 + yC_2 + zC_3$ we get

$$x\Delta = \begin{vmatrix} a_1x+b_1y+c_1z & b_1 & c_1 \\ a_2x+b_2y+c_2z & b_2 & c_2 \\ a_3x+b_3y+c_3z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$\therefore x = \dfrac{\Delta_1}{\Delta}$ where $\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$.

Similarly we get

$y = \dfrac{\Delta_2}{\Delta}$, where $\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $z = \dfrac{\Delta_3}{\Delta}$, where $\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

$\therefore \dfrac{x}{\Delta_1} = \dfrac{y}{\Delta_2} = \dfrac{z}{\Delta_3} = \dfrac{1}{\Delta}$. This is known as **Cramer's Rule.**

3.7.2. Matrix inversion method

Consider the matrix equation $AX = D$, where A is non-singular. Then we can find A^{-1}.

$$AX = D \Leftrightarrow A^{-1}(AX) = A^{-1}D$$
$$\Leftrightarrow (A^{-1}A)\,X = A^{-1}D$$
$$\Leftrightarrow IX = A^{-1}D \quad (\text{I is the unit matrix}).$$
$$\Leftrightarrow X = A^{-1}D$$

From this x, y and z are known.

3.7.3 Solved Problems

1. Problem: *Solve the following simultaneous linear equations by using Cramer's rule.*

$$3x + 4y + 5z = 18$$
$$2x - y + 8z = 13$$
$$5x - 2y + 7z = 20$$

Solution:

Let $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $D = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$

Then we can write the given equations in the form of matrix equation as AX = D.

$$\Delta = \det A = \begin{vmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{vmatrix}$$

$$= 3\begin{vmatrix} -1 & 8 \\ -2 & 7 \end{vmatrix} - 4\begin{vmatrix} 2 & 8 \\ 5 & 7 \end{vmatrix} + 5\begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix}$$

$$= 3(-7+16) - 4(14-40) + 5(-4+5)$$

$$= 27+104+5=136 \neq 0.$$

Hence we can solve the given equation by using Cramer's rule.

$$\Delta_1 = \begin{vmatrix} 18 & 4 & 5 \\ 13 & -1 & 8 \\ 20 & -2 & 7 \end{vmatrix} = 408$$

$$\Delta_2 = \begin{vmatrix} 3 & 18 & 5 \\ 2 & 13 & 8 \\ 5 & 20 & 7 \end{vmatrix} = 136$$

$$\Delta_3 = \begin{vmatrix} 3 & 4 & 18 \\ 2 & -1 & 13 \\ 5 & -2 & 20 \end{vmatrix} = 136.$$

Hence by Cramer's rule,

$$x = \frac{\Delta_1}{\Delta} = \frac{408}{136} = 3; \quad y = \frac{\Delta_2}{\Delta} = \frac{136}{136} = 1 \text{ and } z = \frac{\Delta_3}{\Delta} = \frac{136}{136} = 1.$$

∴ The solution of the given system of equations is $x=3,\ y=1=z$.

2. Problem : *Solve* $3x+4y+5z=18; 2x-y+8z=13$ *and* $5x-2y+7z=20$ *by using 'Matrix inversion method'.*

Solution:

$$\text{Let } A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}; \ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } D = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}.$$

Then we can write the given equations in the form AX = D.

$$\det A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix} = 136 \neq 0.$$

Hence we can solve the given equations by 'Matrix inversion method'.

We have Adj $A = \begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$.

From matrix inversion method,

$$X = A^{-1}D = \frac{\text{Adj}\,A}{\det A}.\ D = \frac{1}{136}\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}\begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore\ x = 3,\ y = 1$ and $z = 1$ is the solution of the given system of equations.

Note

Observe that Cramer's Rule and Matrix inversion method can be applied only when the coefficient matrix A is non-singular. The Gauss-Jordan method given in 3.7.4 below can be applied even otherwise, as in 3.6.13.

3.7.4 Gauss - Jordan method

In this method we try to transform the augmented matrix

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$$

to the form $\begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \end{bmatrix}$ ... (F)

by using elementary row transformations, so that the solution is completely visible that is $x=\alpha,\ y=\beta,\ x=\gamma$. We may get infinitely many solutions or no solution also according to the form of the transformed matrix (F). In fact, this method is an extension of the method already discussed in 3.6.12. The following solved problems (3.7.6) illustrate the method.

3.7.5 Note

For solving a system of three linear equations in three unknowns by Gauss-Jordan method, elementary row operations are performed on the augmented matrix as indicated below.

Step 1

(i) Transform the element in (1,1) position to 1, by a suitable elementary row transformation using the element at (2,1) or (3,1) position or other wise.

(ii) Transform the non-zero elements, if any at (2,1) or (3,1) positions as zeros (other elements of the first column) by using the element 1 at (1,1) position.

If, at the end of step 1, there is a non-zero element at (2,2) or (3,2) position, go to step 2. Otherwise skip it.

Step 2

(i) Transform the element in (2,2) position to 1, by a suitable elementary row transformation using the element at (3,2) position or other wise.

(ii) Transform the non-zero elements, if any, of the second column (i.e., the non-zero elements,if any, at (1,2) or (3, 2) positions) as zeros, by using the element 1 at (2,2) position.

At the end of step 2, or after skipping it for reasons specified above, examine the element at (3,3) position. If it is non zero, go to step 3. Otherwise, stop.

Step 3

(i) Transform the element in (3, 3) position to 1, by dividing R_3 with a suitable number.

(ii) Transform the other non-zero elements if any of the third column (that is, the non-zero elements, if any, at (1,3) or (2, 3) positions) as zeros, by using the 1 present at (3,3) position.

3.7.6 Solved Problems

1. Problem : *Solve the following equations by Gauss - Jordan method*

$$3x+4y+5z=18$$

$$2x-y+8z=13$$

$$5x-2y+7z=20.$$

Solution : The augmented matrix is

$$\begin{bmatrix} 3 & 4 & 5 & 18 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{bmatrix}$$

On applying $R_1 \to R_1 - R_2$ we get

$$\sim \begin{bmatrix} 1 & 5 & -3 & 5 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{bmatrix}$$

On applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 5R_1$, we get

$$\sim \begin{bmatrix} 1 & 5 & -3 & 5 \\ 0 & -11 & 14 & 3 \\ 0 & -27 & 22 & -5 \end{bmatrix}$$

On applying $R_2 \to -5R_2 + 2R_3$, we get

$$\sim \begin{bmatrix} 1 & 5 & -3 & 5 \\ 0 & 1 & -26 & -25 \\ 0 & -27 & 22 & -5 \end{bmatrix}$$

On applying $R_1 \to R_1 - 5R_2, R_3 \to R_3 + 27R_2$, we obtain

$$\sim \begin{bmatrix} 1 & 0 & 127 & 130 \\ 0 & 1 & -126 & -25 \\ 0 & 0 & -680 & -680 \end{bmatrix}$$

On applying $R_3 \to R_3 \div (-680)$, we get

$$\sim \begin{bmatrix} 1 & 0 & 127 & 130 \\ 0 & 1 & -26 & -25 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

On applying $R_1 \to R_1 - 127\ R_3, R_2 \to R_2 + 26R_3$, we get

$$\sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Hence the solution is $x = 3,\ y = 1,\ z = 1$.

2. Problem : *Solve the following system of equations by Gauss - Jordan method*

$x + y + z = 3,\ 2x + 2y - z = 3, x + y - z = 1$.

Solution : The matrix equation is $AX = D$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix};\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } D = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}.$$

The augmented matrix is

$$[AD] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & -1 & 3 \\ 1 & 1 & -1 & 1 \end{bmatrix}.$$

On applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$, we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -2 & -2 \end{bmatrix}.$$

On applying $R_3 \to R_3 - \left(\frac{2}{3}\right)R_2$, we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence the following is the system of equations equivalent to the given system of equations.

$$x + y + z = 3$$

$$-3z = -3.$$

Hence $z = 1, x + y = 2.$

$\therefore$ The solution set is

$x = k,\ y = 2 - k,\ z = 1,$ where $k \in \mathbf{R}$.

3. Problem : *By using Gauss-Jordan method, show that the following system has no solution*

$$2x + 4y - z = 0,\ x + 2y + 2z = 5,\quad 3x + 6y - 7z = 2.$$

Solution : The equivalent matrix equation is AX = D, where

$$A = \begin{bmatrix} 2 & 4 & -1 \\ 1 & 2 & 2 \\ 3 & 6 & -7 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}.$$

The augmented matrix is

$$[A\ D] = \begin{bmatrix} 2 & 4 & -1 & 0 \\ 1 & 2 & 2 & 5 \\ 3 & 6 & -7 & 2 \end{bmatrix}.$$

On interchanging R_1 and R_2 we get

$$\sim \begin{bmatrix} 1 & 2 & 2 & 5 \\ 2 & 4 & -1 & 0 \\ 3 & 6 & -7 & 2 \end{bmatrix}.$$

On applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1,$ we get

$$\sim \begin{bmatrix} 1 & 2 & 2 & 5 \\ 0 & 0 & -5 & -10 \\ 0 & 0 & -13 & -13 \end{bmatrix}.$$

On applying $R_2 \to R_2 \div (-5),\ R_3 \to R_3 \div (-13),$ we get

$$\sim \begin{bmatrix} 1 & 2 & 2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

On applying $R_3 \to R_3 - R_2,$ we get

$$\sim \begin{bmatrix} 1 & 2 & 2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Hence the given system of equations is equivalent to the following system of equations

$$x+2y+2z=5,\ z=2, 0(x)+0(y)+0(z)=-1.$$

Clearly no x, y, z satisfy the last equation in the above system.

Hence the given system has no solution.

Exercise 3 (h)

I. Solve the following systems of equations.

(i) by using Cramer's rule and Matrix inversion method, when the coefficient matrix is non-singular.

(ii) by using Gauss-Jordan method. Also determine whether the system has a unique solution or infinite number of solutions or no solution and find the solutions if exist.

1. $5x-6y+4z=15$
$7x+4y-3z=19$
$2x+y+6z=46$

2. $x+y+z=1$
$2x+2y+3z=6$
$x+4y+9z=3$

3. $x-y+3z=5$
$4x+2y-z=0$
$-x+3y+z=5$

4. $2x+6y+11=0$
$6x+20y-6z+3=0$
$6y-18z+1=0$

5. $2x-y+3z=9$
$x+y+z=6$
$x-y+z=2$

6. $2x-y+8z=13$
$3x+4y+5z=18$
$5x-2y+7z=20$

7. $2x-y+3z=8$
$-x+2y+z=4$
$3x+y-4z=0$

8. $x+y+z=9$
$2x+5y+7z=52$
$2x+y-z=0$

3.7.7 Solution of a homogeneous system of linear equations

We consider the following homogeneous linear equations

$$a_1x+b_1y+c_1z=0$$
$$a_2x+b_2y+c_2z=0$$
$$a_3x+b_3y+c_3z=0\ .$$

The equivalent matrix equation of the above system is $AX=O$ where

$$A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix},\ X=\begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } O=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Clearly the coefficient matrix A and the augmented matrix have the same rank, for they differ by a column of zeros. Thus a system of homogeneous equations is always consistent. In fact, $x = y = z = 0$ is always a solution. We call this the **trivial solution**. We are however, interested in finding whether or not there are non trivial solutions.

We state below a theorem without proof, which indicates the nature of solutions of the system.

3.7.8 Theorem

The system of equations $AX = O$ *has*

(i) *the trivial solution only, if rank (A) is* 3

(ii) *an infinite number of solutions if rank (A) is less than* 3.

The method of solving a system of homogeneous linear equations is similar to that adopted on the examples given in 3.6.13. However, some problems are solved here under.

3.7.9 Solved Problems

1. Problem: *Find the non-trivial solutions, if any, for the following system of equations.*

$$2x+5y+6z=0$$
$$x-3y+8z \;=0$$
$$3x+y-4z \;=0\cdot$$

Solution : The coefficient matrix $A = \begin{bmatrix} 2 & 5 & 6 \\ 1 & -3 & -8 \\ 3 & 1 & -4 \end{bmatrix}$.

On interchanging R_1 and R_2 we get

$$A \sim \begin{bmatrix} 1 & -3 & -8 \\ 2 & 5 & 6 \\ 3 & 1 & -4 \end{bmatrix}$$

On applying $R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1,$ we get

$$A \sim \begin{bmatrix} 1 & -3 & -8 \\ 0 & 11 & 22 \\ 0 & 10 & 20 \end{bmatrix}.$$

On applying $R_2 \to R_2 - R_3,$ we get

$$A \sim \begin{bmatrix} 1 & -3 & -8 \\ 0 & 1 & 2 \\ 0 & 10 & 20 \end{bmatrix}.$$

On applying $R_3 \to R_3 \div 10,$ we get

$$A \sim \begin{bmatrix} 1 & -3 & -8 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad \ldots \text{(F)}$$

det A = 0 as R_2 and R_3 are identical.

Clearly rank (A) = 2, as the sub matrix $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$ is non- singular.

Hence the system has non-trivial solution.

Writing back the equations from (F)

$$x - 3y - 8z = 0$$
$$y + 2z = 0$$

On giving an arbitrary value k to z, we obtain the solution set as

$x = 2k,\ y = -2k,\ z = k,\ k \in \mathbf{R}$. For $k \neq 0$ we get non-trivial solutions.

2. Problem: *Find whether the following system of linear homogeneous equations has a non-trivial solution.*

$$x - y + z = 0$$
$$x + 2y - z = 0$$
$$2x + y + 3z = 0$$

Solution : The coefficient matrix is $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$.

Its determinant is 9. Hence the system has the trivial solution $x = y = z = 0$ only.

Exercise 3 (i)

Solve the following systems of homogeneous equations.

1. $2x + 3y - z = 0$
$x - y - 2z = 0$
$3x + y + 3z = 0$

2. $3x + y - 2z = 0$
$x + y + z = 0$
$x - 2y + z = 0$

3. $x + y - 2z = 0$
$2x + y - 3z = 0$
$5x + 4y - 9z = 0$

4. $x + y - z = 0$
$x - 2y + z = 0$
$3x + 6y - 5z = 0$

Key Concepts

- An $m \times n$ matrix A is represented as $A = [a_{ij}]_{m \times n}$.
- A matrix is called square matrix if its number of rows equals number of columns.
  An element a_{ij} is in principal diagonal if $i = j$.
  The sum of the elements of the Principal diagonal is called Trace of the matrix.
- A square matrix is called a
  (i) Diagonal matrix if each non-diagonal element is zero.
  (ii) Scalar matrix if each non-diagonal element is zero and every diagonal element is equal to some scalar k.
  (iii) Unit matrix or Identity matrix if each non-diagonal element is zero and each diagonal element is equal to 1.
- If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ then $A + B = [c_{ij}]_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij}$.
- If $A = [a_{ij}]_{m \times n}$ and k is a scalar, then $kA = [ka_{ij}]_{m \times n}$.
- If $A = [a_{ik}]_{m \times n}$ and $B = [b_{kj}]_{n \times p}$ then $AB = [c_{ij}]_{m \times p}$ where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.
- The matrix obtained by interchanging rows and columns is called Transpose of the given matrix. Transpose of A is denoted by A' or A^T.
- A matrix is called
  (i) Symmetric if $A' = A$
  (ii) Skew-symmetric if $A' = -A$.
- A matrix obtained by deleting some rows or columns (or both) of a matrix is called a submatrix of the given matrix.
- Let A be a 3×3 matrix. Then
  (i) The minor of an element is the determinant of the 2×2 sub matrix obtained by deleting the row and column in which the element is present
  (ii) the cofactor of an element a_{ij} is the product of its minor and $(-1)^{i+j}$
  (iii) the determinant of A is the sum of the products of the elements of any row (or column) with the corresponding cofactors.
- A square matrix is said to be
  (i) singular if its determinant is zero
  (ii) non-singular otherwise.
- Adjoint of a square matrix A (order >1) is the transpose of the matrix formed by replacing the elements by cofactors.
- Let A be a square matrix. A matrix B, if exists, such that AB = BA = I is called the inverse of A and is denoted by A^{-1}.

❖ The rank of a non-zero matrix A is defined as the maximum of the order of the non-singular square submatrices of A. The rank of a null matrix is defined as zero. The rank of a matrix A is denoted as rank (A). In particular, If A is a 3×3 matrix, then its rank is

(i) 3 if A is non-singular

(ii) 2 if A is singular and atleast one of its 2×2 sub matrices is non-singular

(iii) 1 if every 2×2 sub matrix is singular.

❖ The following transformations are known as elementary transformations on a matrix.

(i) Interchange of two rows (or columns)

(ii) Multiplication of the elements of a row (or column) by a non-zero number

(iii) Addition to the elements of a row (or a column), the corresponding elements of another row (or column) multiplied by any number.

❖ Elementary transformations on a matrix do not change its rank.

❖ A system of linear equations is

(i) consistent if it has a solution

(ii) inconsistent if it has no solution.

❖ Non-homogeneous system

$$a_1x + b_1 y + c_1z = d_1$$
$$a_2x + b_2 y + c_2z = d_2$$
$$a_3x + b_3 y + c_3z = d_3$$

The above system of equations has

(i) a unique solution if rank (A) = rank ([A D]) = 3

(ii) infinitely many solutions if rank (A) = rank ([A D]) < 3.

(iii) no solution if rank (A) $\neq$ rank ([A D]).

❖ Homogeneous system of equations

$$a_1x + b_1 y + c_1z = 0$$
$$a_2x + b_2 y + c_2z = 0$$
$$a_3x + b_3 y + c_3z = 0 \; .$$

The above system has

(i) Trivial solution $x = y = z = 0$ only if rank (A) = 3

(ii) infinitely many non-trivial solutions if rank (A) < 3.

Historical Note

The history of matrices and determinants goes back to the second century B.C. although traces can be seen as early as the fourth century B.C. However it was not until near the end of the 17th century that the ideas reappeared and development really got underway.

It is not surprising that the beginnings of matrices and determinants should arise through the study of systems of linear equations. The Babylonians studied the problems which led to simultaneous linear equations and some of these are preserved in clay tablets which still survive. The Chinese, between 200 B.C. and 100 B.C. came much closer to matrices than the Babylonians. The text **Nine Chapters of the Mathematical Art** (*Chiu Chang Suan Shu*) written during the Han dynasty gives the first known example of matrix methods to solve simultaneous equations.

The rectangular arrangements of certain numbers in some rows and columns was named as "Matrix" by *J.J. Sylvester* in 1850. *Arthur Cayley* (1821-1895), an English mathematician, is also known for his matrix representation of simultaneous equations.

Since their first appearance in ancient China, Matrices have remained as important mathematical tools. Matrix theory is used as an indispensible tool in the study of Physical Sciences, Engineering, Statistics, Economics, Sociology etc. Today they are used, not simply for solving systems of simultaneous linear equations, but also for describing ***Quantum mechanics*** of atomic structure, designing ***computer game graphics***, analysing ***relationships*** and even plotting complicated ***dance steps!***

The elevation of the knowledge of matrix from a mere tool to an important mathematical theory owes a lot to the work of a lady mathematician, *Olga Taussky Todd* (1906-1995), who began by using matrices to analyse vibrations on airplanes during World War II and became the torch bearer for matix theory.

Matrices are indispensable in some applications and models in other branches of mathematics. Some of the various types of matrices are Symmetric, Hermitian, Triangular, Diagonal, Tridiagonal, Band-centro symmetric, Toeplitz, Positive definite Hessian, Circulant and so on.....

Answers

Exercise 3(a)

I. 1. (i) $[2 \quad 1 \quad 3]$ (ii) $\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 7 & 9 & 2 \\ 8 & 9 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} -1 & 3 \\ 0 & -2 \\ 1 & 0 \end{bmatrix}$

2. $x_1 = 1,\ x_2 = 4,\ x_3 = 7,\ x_4 = -3$ **3.** $\begin{bmatrix} -2 & -3 & 10 \\ 2 & 1 & 8 \\ 5 & 1 & 1 \end{bmatrix}$

4. $X = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -1 & 3 \\ 5 & 2 & 3 \end{bmatrix}$

5. $x = 8,\ y = 5,\ z = -4,\ a = 10$

II. 1. $x = 2,\ y = 2,\ z = 5,\ a = 5$

2. 1

3. $\begin{bmatrix} -1 & 1 & 1 \\ -2 & -2 & -4 \\ -4 & -5 & 5 \end{bmatrix}, \begin{bmatrix} 5 & -6 & -7 \\ 8 & 7 & 16 \\ 16 & 20 & -19 \end{bmatrix}$

4. $\begin{bmatrix} 7 & 2 & -3 \\ -3 & 2 & 7 \end{bmatrix}$

Exercise 3(b)

I. 1. (i) [5]

(ii) $\begin{bmatrix} 8 \\ 5 \end{bmatrix}$

(iii) $\begin{bmatrix} 8 & -13 \\ 16 & 29 \end{bmatrix}$

(iv) I_{3x3}

(v) Not possible (vi) Not possible (vii) O_{2x2} (viii) O_{3x3}

2. AB and BA exist; $AB = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$, $BA = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$ A and B are not commutative.

3. $\begin{bmatrix} 14 & 10 \\ -5 & -1 \end{bmatrix}$

4. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

6. $AB = \begin{bmatrix} 7 & 4 & 4 \\ 6 & 2 & 12 \end{bmatrix}$; BA does not exist.

7. -2

II. 1. $A^4 = (3I)^4 = 81\,I$

2. O_{3x3}

3. O_{3x3}

III. 4. $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

(This is just one example satisfying the given conditions)

5. (a) 15,000 and 15,000 (b) 5,000 and 25,000.

Exercise 3(c)

I. 1. $\begin{bmatrix} -2 & 2 \\ -2 & -9 \end{bmatrix}$

2. $\begin{bmatrix} -6 & 6 \\ 13 & 0 \\ -1 & 10 \end{bmatrix}, \begin{bmatrix} -4 & 11 \\ 4 & 0 \\ 4 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 4 & -9 \\ -9 & 6 \end{bmatrix}, \begin{bmatrix} 20 & -22 \\ -22 & 34 \end{bmatrix}$

4. 6

5. 2

6. Skew-symmetric

II. 2. $\begin{bmatrix} -5 & 15 & 5 \\ 10 & 20 & -8 \\ 9 & -23 & -15 \end{bmatrix}$ **3.** $\begin{bmatrix} -12 & 24 & -7 \\ 0 & 0 & 1 \\ -13 & 26 & -5 \end{bmatrix}, \begin{bmatrix} -12 & 0 & -13 \\ 24 & 0 & 26 \\ -7 & 1 & -5 \end{bmatrix}$

Exercise 3(d)

I. 1. **(i)** -11 **(ii)** 38 **(iii)** 1 **(iv)** 2 **(v)** -108 **(vi)** 37 **(vii)** 0

(viii) $abc + 2fgh - af^2 - bg^2 - ch^2$

(ix) $3abc - a^3 - b^3 - c^3$ **(x)** -8 **2.** 7

II. **6.** $a_1 b_2 c_3$.

Exercise 3(e)

1. **(i)** $\begin{bmatrix} 6 & 3 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{-1}{6} & \frac{1}{12} \end{bmatrix}$ **(ii)** $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}, \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}$ **(iv)** $\begin{bmatrix} -2 & 3 & 1 \\ 1 & -2 & 0 \\ 2 & -2 & -1 \end{bmatrix}, \begin{bmatrix} -2 & 3 & 1 \\ 1 & -2 & 0 \\ 2 & -2 & -1 \end{bmatrix}$

2. $\begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$ **3.** $\begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$

4. $\frac{1}{9}\begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$ **5.** $\begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$

Exercise 3(f)

I.	**1.** 1	**2.** 2	**3.** 1	**4.** 2	**5.** 2	**6.** 2
II.	**1.** 3	**2.** 3	**3.** 2	**4.** 1	**5.** 3	**6.** 3

Exercise 3(g)

I. 1. Inconsistent, no solution.

2. Consistent; Unique solution ; $x = 1, y = 2, z = 3$.

3. Consistent; Infinitely many solutions ;
solutions set = $\{ (x, y, z) : x = 1, y + z = 0\}$.

4. Consistent; Unique solution ; $x = 1, y = 3, z = 5$.

5. Consistent; Unique solution ; $x = -7, y = 22, z = -9$.

6. Consistent; Infinitely many solutions ; $x = -1 + 2k, \; y = 3 - 2k, \; z = k$; k is a scalar.

7. Consistent; Unique solution ; $x = \frac{35}{18}, \; y = \frac{29}{18}, \; z = \frac{5}{18}$

8. Consistent; Unique solution ; $x = 2, \; y = 2, \; z = \frac{1}{2}$.

Exercise 3(h)

I. 1. $x = 3, y = 4, z = 6$; Unique solution

2. $x = 7, y = -10, z = 4$; Unique solution

3. $x = 0, y = 1, z = 2$; Unique solution

4. No solution

5. $x = 1, y = 2, z = 3$; Unique solution

6. $x = 3, y = 1, z = 1$; Unique solution

7. $x = y = z = 2$; Unique solution

8. $x = 1, y = 3, z = 5$; Unique solution

Exercise 3(i)

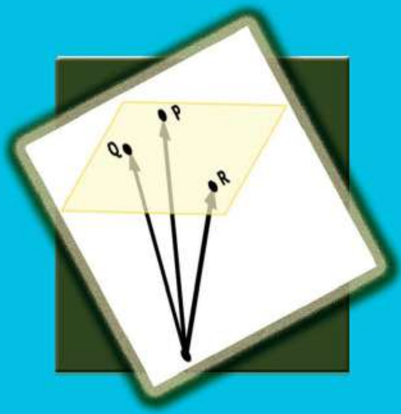
I. 1. $x = y = z = 0$

2. $x = y = z = 0$

3. $x = y = z = k$ for any real number k

4. $x = k, \; y = 2k, \; z = 3k$ for any real number k

Vector Algebra



Chapter 4

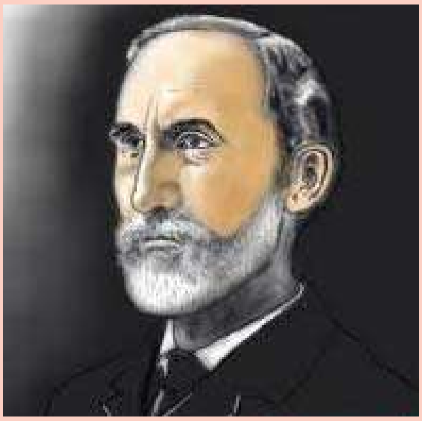
Addition of Vectors

"Vectors are not merely a pretty toy suitable only for elegant proof of general theorems, but are a powerful weapon of work away on mathematical investigation, both in research and in solving problems"

– Chapman

Introduction

In our day to day life we come across many queries such as - What is your height? How should a foot ball player hit the ball, to give a pass to one another player of his team? Observe that one possible answer to the first query is 1.7 meters, a quantity that specifies a value (magnitude) which is a real number. Such quantities are called scalars. However, the answer to the second query is a quantity (called force) which involves muscular strength (magnitude) and also direction (in which another player is positioned). Such quantities are called vectors. In Physics, Engineering and Mathematics, we frequently come across with both types of quantities, namely scalar quantities such as length, mass, volume, temperature, density, area, work, resistance etc. and vector quantities like displacement, velocity, acceleration, force, weight, momentum etc.



Josiah Willard Gibbs

(1839 - 1903)

Gibbs was a prominent American engineer and promoter of vector analysis, which established itself as a more easily applied subject compared to Hamilton's quaternions or Grassmann's Calculus of extensions.

Vector methods have revolutionised Mechanics, Engineering, Physics and Mathematics. Rene Descarte (1596-1660), after whom the Cartesian coordinate system is named, G.W. Leibnitz (1646-1716), a famous mathematician of 17th century and R.Hamilton (1805-1865), a well known theoretical physicist are the trio who laid the seeds to this branch of Mathematics. J.W. Gibb's (1839-1903) work on vector analysis was of major importance in Mathematics.

In this chapter, we will study some of the basic concepts about vectors, various operations on vectors and their algebraic and geometric properties. Angle between two non-zero vectors, linear combination of vectors, vector equations of line and plane are discussed to give a full realisation of the applicability of vectors in various areas as mentioned above.

4.1 Vectors as a triad of real numbers, some basic concepts

Let l be any straight line in a plane or three dimensional space. This line can be given two directions by means of arrow heads. A line with one of these directions prescribed, is called a directed line (Fig. 4.1).

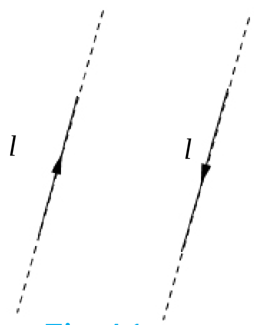


Fig. 4.1

4.1.1 Definition : (Directed line segment)

If A and B are two distinct points in the space, the ordered pair (A, B), denoted by AB is called a directed line segment with initial point A and terminal point B.

The magnitude of AB, denoted by |AB| = a (say), is the length of AB or the distance between A and B (Fig. 4.2).

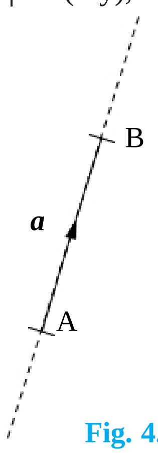


Fig. 4.2

4.1.2 Definition

A line segment with a specified magnitude and direction is called a vector.

Notice that the directed line segment in Fig. 4.2 is a vector denoted by $\overrightarrow{AB}$ or **AB** or simply as ***a*** and read as vector **AB** or vector ***a***. The arrow indicates the direction of the vector. When A $\neq$ B, we say that the line $\overleftrightarrow{AB}$ is the support of $\overline{AB}$.

The **zero vector**, denoted by **0**, is the collection of **PP**, where P is any point in the space. The zero vector, also known as the **null vector**, has neither support nor any specific direction. Observe that, for the zero vector, the initial and terminal points coincide and its magnitude is the scalar 0.

Let *a*, *b* and *c* be real numbers (not necessarily distinct). A set formed with *a*, *b*, *c* in which the order of occurance is also preassigned is called **an ordered triad** or **a triple**. If *a*, *b*, *c* are distinct reals, then we get six ordered triads, namely (*a*, *b*, *c*), (*b*, *c*, *a*), (*c*, *a*, *b*) etc. For the ordered traid (*a*, *b*, *c*), *a*, *b*, *c* are called **the first**, **the second** and **the third** components respectively.

The set of all ordered triads (*a*, *b*, *c*) of real numbers is denoted by $\mathbb{R}^3$. This representation will be used in rectangular coordinate system in section 4.7.2.

4.1.3 Position Vector

Consider a three - dimensional rectangular coordinate system OX, OY, OZ and a point P in the space having coordinates (*x*, *y*, *z*) with respect to the origin O(0, 0, 0) as shown in Fig. 4.3(a). Then the vector **OP** having O and P as its initial and terminal points respectively, is called the position vector of the point P with respect to O. This is denoted by ***r***. Then the magnitude of **OP**, using the distance formula, is given by

$$|\overrightarrow{OP}| = |\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2}\,.$$

It is customary that the position vector of a point A, with respect to the origin O is denoted by ***a*** (Fig. 4.3(b)).

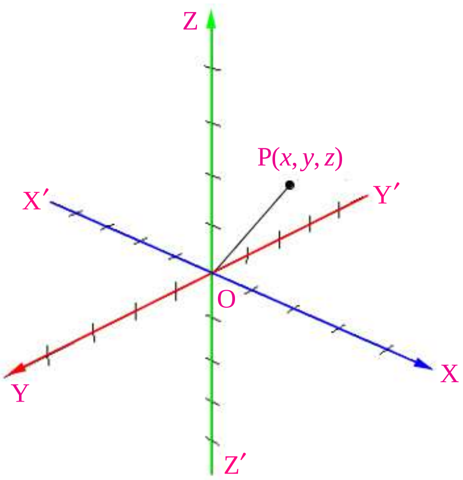


Fig. 4.3(a)

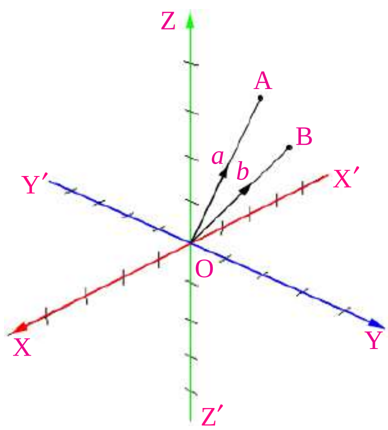


Fig. 4.3(b)

4.1.4 Direction cosines and Direction ratios

Consider the position vector **OP** = ***r*** of a point P(x, y, z). Let α, β, γ be the angles made by the vector ***r*** with the positive direction (counter clockwise direction) of X, Y, Z axes respectively. Then cos α, cos β and cos γ are called the direction cosines of the vector ***r***. These direction cosines are usually denoted by l, m, n respectively.

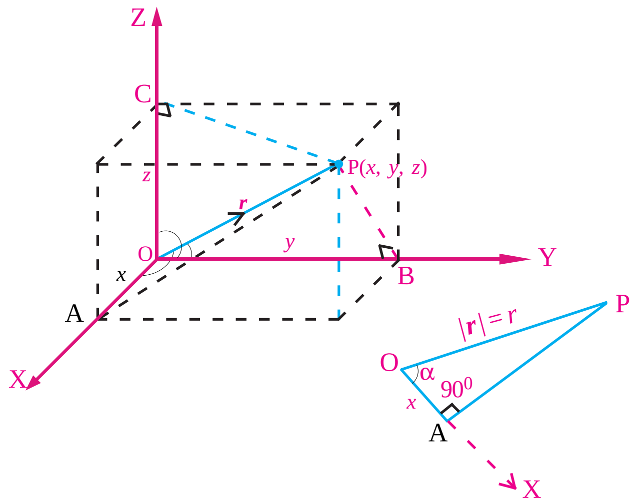


Fig. 4.4

Draw perpendiculars from P to the X, Y and Z axes and let A, B, C be the feet of the perpendiculars respectively (See Fig. 4.4).

From Fig. 4.4, we observe that ΔOAP is right angled and hence $\cos\alpha = \dfrac{x}{|\boldsymbol{r}| = r}$. Similarly from the right angled triangles OBP and OCP, we may write $\cos\beta = \dfrac{y}{r}$ and $\cos\gamma = \dfrac{z}{r}$. Thus the coordinates x, y, z of the point P may also be expressed as (lr, mr, nr). The numbers lr, mr, nr which are proportional to the direction cosines l, m, n are called the **direction ratios** of the vector ***r***. These are usually denoted by a, b, c respectively.

We observe here that

$$r^2 = x^2 + y^2 + z^2$$

$$= l^2r^2 + m^2r^2 + n^2r^2$$

$$= r^2(l^2 + m^2 + n^2)$$

so that $l^2 + m^2 + n^2 = 1$ but $a^2 + b^2 + c^2 \neq 1$ in general.

4.2 Classification (Types) of vectors

4.2.1 Definition (Unit vector)

A vector whose magnitude is unity (i.e., 1 unit) is called a unit vector. It is represented by $\boldsymbol{e}$*. The unit vector in the direction of a given vector* $\boldsymbol{a}$ *is usually denoted by* $\hat{\boldsymbol{a}}$.

4.2.2 Definition (Equal vectors)

Two vectors $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are said to be equal and written as* $\boldsymbol{a} = \boldsymbol{b}$*, if they have the same magnitude and direction, regardless of the positions of their initial points.*

4.2.3 Definition (Collinear vectors, like and unlike vectors)

Two or more vectors are said to be collinear if they are parallel to the same line, irrespective of their magnitudes and direction. Such vectors have the same support or parallel support.

Two vectors are called like or unlike vectors according as they have the same direction or opposite direction. In the following figure (Fig. 4.5) $\boldsymbol{a}$ and $\boldsymbol{b}$ are like vectors, where as $\boldsymbol{a}$ and $\boldsymbol{c}$ are unlike vectors.

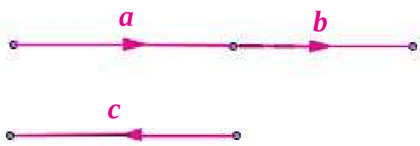


Fig. 4.5

4.2.4 Negative of a vector

Let $\boldsymbol{a}$ be a vector. The vector having the same magnitude as $\boldsymbol{a}$ but having the opposite direction is called the negative vector of $\boldsymbol{a}$ and is denoted by $-\boldsymbol{a}$. Note that if $\boldsymbol{a} = \mathbf{AB}$ then $-\boldsymbol{a} = \mathbf{BA}$.

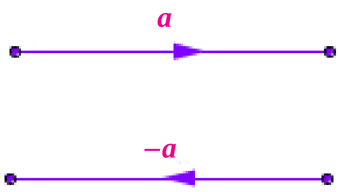


Fig. 4.6

Note : Co-initial vectors : Two or more vectors having the same initial point are called **co-initial vectors**.

4.2.5 Definition (Coplanar vectors)

Vectors whose supports are in the same plane or parallel to the same plane are called coplanar vectors. Vectors which are not coplanar are called non-coplanar vectors.

Note that the vectors $\boldsymbol{a}$ = **PA**, $\boldsymbol{b}$ = **PB** and $\boldsymbol{c}$ = **PC** are coplanar vectors if and only if the four points P, A, B, C lie in the same plane. Coplanarity or non coplanarity of vectors arises only when there are three or more non-zero vectors, since any two vectors are always coplanar.

4.3 Sum (Addition) of vectors

We shall now introduce the concept of addition (sum) of vectors, derive the commutative law, associative law and a few other properties.

4.3.1 Triangle law of vector addition

A vector **AB** simply means the displacement from a point A to the point B along the line AB. Now consider a situation that a person moves from A to B and then from B to C (Fig. 4.7). The net displacement made by the person from point A to the point C, is given by the vector **AC** and expressed as

AC = AB + BC

This is known as the *triangle law of vector addition*.

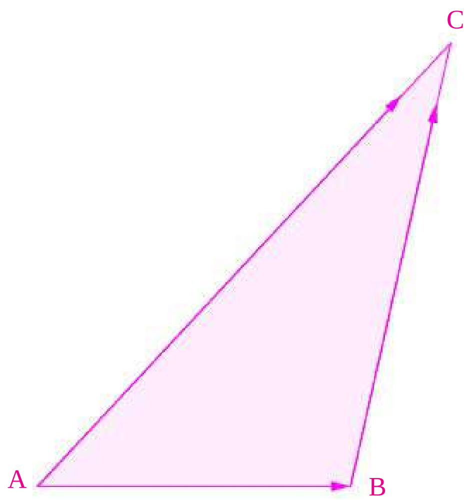


Fig. 4.7

In general, if we have two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (Fig. 4.8(i)), then to add them, they are positioned, so that the initial point of one coincides with the terminal point of the other (Fig. 4.8(ii)).

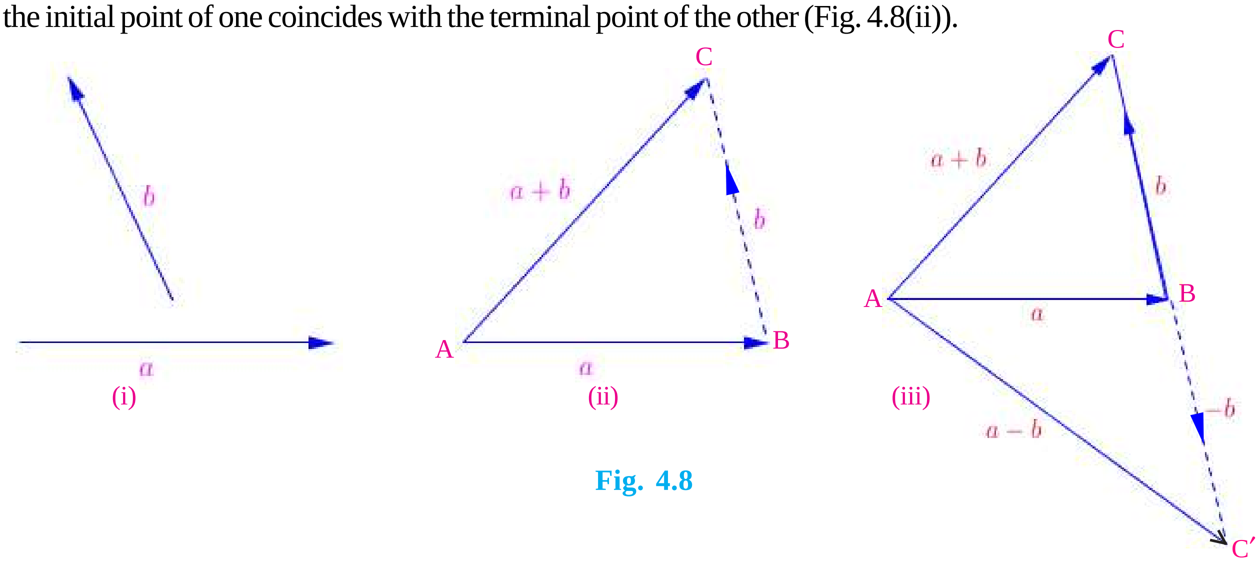


Fig. 4.8

For example, in Fig. 4.8(ii), we have shifted vector $\boldsymbol{b}$ without changing its magnitude and direction, so that it's initial point coincides with the terminal point of $\boldsymbol{a}$. Then, the vector $\boldsymbol{a} + \boldsymbol{b}$, represented by the third side AC of the triangle ABC, gives us the sum (or resultant) of the vectors ***a*** and ***b*** i.e., in triangle ABC (Fig. 4.8(ii)), we have

$$\mathbf{AB} + \mathbf{BC} = \mathbf{AC}$$

Now, again since $\mathbf{AC} = -\mathbf{CA}$, from the above equation, we have

$$\mathbf{AB} + \mathbf{BC} + \mathbf{CA} = \mathbf{AA} = \mathbf{0}.$$

This means that when the sides of a triangle are taken in order, it leads to zero resultant as the initial and terminal points get coincided (Fig. 4.8(iii)).

Now, construct the vector $\mathbf{BC'}$ so that its magnitude is the same as the vector **BC**, in the direction opposite to that of it (Fig. 4.8(iii)), i.e., $\mathbf{BC'} = -\mathbf{BC}$.

Then, on applying triangle law, from Fig. 4.8(iii), we have

$$\mathbf{AC'} = \mathbf{AB} + \mathbf{BC'} = \mathbf{AB} + (-\mathbf{BC}) = \boldsymbol{a} - \boldsymbol{b}.$$

The vector $\mathbf{AC'}$ is said to represent the difference of ***a*** and ***b***.

4.3.2 Parallelogram law of vector addition

Now, consider a boat in a river going from one bank of the river to the other in a direction perpendicular to the flow of the river. Then, it is acted upon by two velocity vectors - one is the velocity imparted to the boat by its engine and the other is the velocity of the flow of river water. Under the simultaneous influence of these two velocities, the boat actually starts travelling with a different velocity. To have a precise idea about the effective speed and direction (i.e., the resultant velocity) of the boat, we have the following law of vector addition.

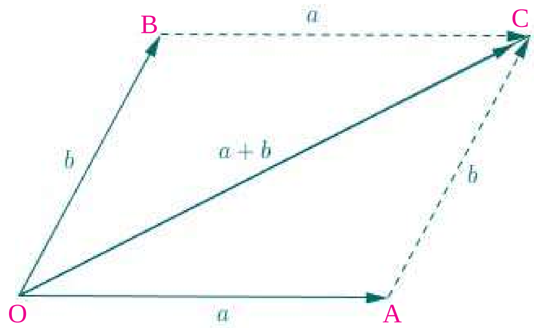


Fig. 4.9

If we have two vectors ***a*** and ***b*** represented by the two adjacent sides of a parallelogram in magnitude and direction (Fig. 4.9), then their sum ***a*** + ***b*** is represented in magnitude and direction by the diagonal of the parallelogram through their common point. This is known as the ***parallelogram law of vector addition***.

Note : From Fig. 4.9, using the triangle law, one may note that

$$\mathbf{OA} + \mathbf{AC} = \mathbf{OC}$$

or $$\mathbf{OA} + \mathbf{OB} = \mathbf{OC} \quad (\text{since } \mathbf{AC} = \mathbf{OB})$$

which is parallelogram law. Thus, we may say that the two laws of vector addition are equivalent to each other.

4.3.3 Properties of vector addition

1. Property : For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a} \text{ (commutative property).}$$

Proof : Consider the parallelogram ABCD (Fig. 4.10). Let **AB** = $\boldsymbol{a}$ and **BC** = $\boldsymbol{b}$, then using the triangle law, for triangle ABC, we have

$$\mathbf{AC} = \boldsymbol{a} + \boldsymbol{b}.$$

Now, since the opposite sides of a parallelogram are equal and parallel, from Fig. 4.10, we have **AD** = **BC** = $\boldsymbol{b}$ and **DC** = **AB** = $\boldsymbol{a}$. Again using triangle law, for triangle ADC, we have

$$\mathbf{AC} = \mathbf{AD} + \mathbf{DC} = \boldsymbol{b} + \boldsymbol{a}.$$

Hence $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$.

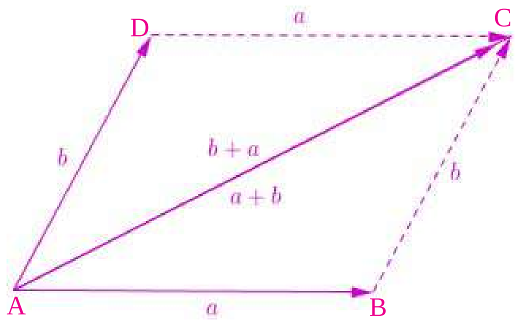
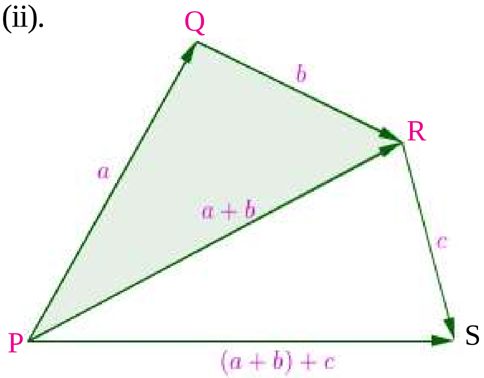


Fig. 4.10

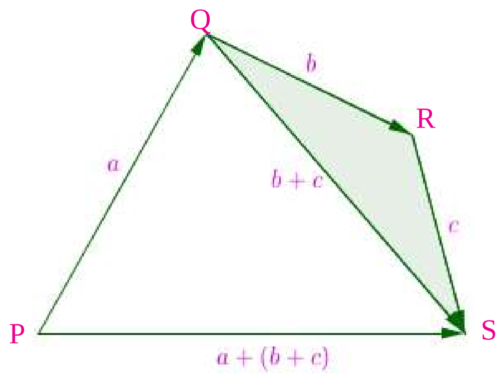
2. Property : For any three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$

$$(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) \qquad \text{(Associative property)}$$

Proof: Let the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be represented by **PQ**, **QR** and **RS** respectively, as shown in Fig. 4.11(i) and (ii).



(i)



(ii)

Fig. 4.11

Then $\boldsymbol{a} + \boldsymbol{b} = \mathbf{PQ} + \mathbf{QR} = \mathbf{PR}$

and $\boldsymbol{b} + \boldsymbol{c} = \mathbf{QR} + \mathbf{RS} = \mathbf{QS}$

so $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \mathbf{PR} + \mathbf{RS} = \mathbf{PS}$

and $\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = \mathbf{PQ} + \mathbf{QS} = \mathbf{PS}$

Hence $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$.

Remark : The associative property of vector addition enables us to write the sum of three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ as $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$ without using brackets.

3. Property : For any vector $\boldsymbol{a}$, $\boldsymbol{a} + \mathbf{0} = \mathbf{0} + \boldsymbol{a} = \boldsymbol{a}$

We have $\boldsymbol{a} + \mathbf{0} = \mathbf{PQ} + \mathbf{QQ} = \mathbf{PQ} = \boldsymbol{a}$

$\therefore \ \boldsymbol{a} + \mathbf{0} = \mathbf{0} + \boldsymbol{a} = \boldsymbol{a}$, by property (1).

Here, the zero vector **0** is called the additive identity for the vector addition.

We know that for any two real numbers x and y, $|x+y| \le |x| + |y|$ and $|x-y| > ||x| - |y||$. We shall now establish similar properties for the magnitudes of the vectors.

4.3.4 Theorem: *Let* $\boldsymbol{a}, \boldsymbol{b}$ *be two vectors. Then*

(i) $|\boldsymbol{a}+\boldsymbol{b}| \le |\boldsymbol{a}| + |\boldsymbol{b}|$ (*equality holds if and only if* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are like vectors*).

(ii) $|\boldsymbol{a}-\boldsymbol{b}| \ge ||\boldsymbol{a}| - |\boldsymbol{b}||$ (*equality holds if and only if* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are like vectors.*

Proof

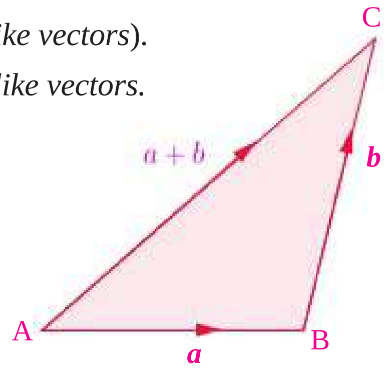


Fig. 4.12

(i) Choose points A, B and C such that $\mathbf{AB} = \boldsymbol{a}$ and $\mathbf{BC} = \boldsymbol{b}$

(see Fig. 4.12). Then

$$|\boldsymbol{a}+\boldsymbol{b}| = AC \le AB + BC = |\boldsymbol{a}| + |\boldsymbol{b}|.$$

The equality holds if and only if B belongs to the line segment AC, that is $\boldsymbol{a}$ and $\boldsymbol{b}$ are like vectors.

(ii) $|\boldsymbol{a}| = |(\boldsymbol{a}-\boldsymbol{b}) + \boldsymbol{b}| \le |\boldsymbol{a}-\boldsymbol{b}| + |\boldsymbol{b}|$

$$\therefore \ |\boldsymbol{a}| - |\boldsymbol{b}| \le |\boldsymbol{a}-\boldsymbol{b}| \qquad \text{... (1)}$$

Equality $|\boldsymbol{a}| - |\boldsymbol{b}| = |\boldsymbol{a}-\boldsymbol{b}|$ takes place if and only if the vectors $\boldsymbol{b}$ and $(\boldsymbol{a}-\boldsymbol{b})$ are like vectors and hence $\boldsymbol{b}$ and $(\boldsymbol{a}-\boldsymbol{b}) + \boldsymbol{b} = \boldsymbol{a}$ are like vectors.

Thus $|\boldsymbol{a}| - |\boldsymbol{b}| \le |\boldsymbol{a}-\boldsymbol{b}|$ (equality if and only if $\boldsymbol{a}, \boldsymbol{b}$ are like vectors). ... (2)

Similarly,

$|\boldsymbol{b}| - |\boldsymbol{a}| \le |\boldsymbol{b}-\boldsymbol{a}| = |\boldsymbol{a}-\boldsymbol{b}|$ (with equality if and only if $\boldsymbol{a}$ and $\boldsymbol{b}$ are like vectors). ... (3)

Combining (2) and (3) we get that $||\boldsymbol{a}| - |\boldsymbol{b}|| \le |\boldsymbol{a}-\boldsymbol{b}|$ with equality if and only if $\boldsymbol{a}$ and $\boldsymbol{b}$ are like vectors.

4.4 Scalar Multiplication of a vector

We shall now introduce the operation of scalar multiplication of a vector, initially through a geometric visualisation and later state some laws of scalar multiplication.

4.4.1 Scalar multiplication : Let $\boldsymbol{a}$ be a given non zero vector and λ a scalar. Then the product of the vector $\boldsymbol{a}$ by the scalar λ, denoted as $\lambda\boldsymbol{a}$, is defined as a vector $\lambda\boldsymbol{a}$ collinear with $\boldsymbol{a}$. The vector $\lambda\boldsymbol{a}$ is called the multiplication of vector $\boldsymbol{a}$ by the scalar λ and $\lambda\boldsymbol{a}$ has the direction same (or opposite) to that of vector $\boldsymbol{a}$ according as the value of λ is positive (or negative). Also, the magnitude of vector $\lambda\boldsymbol{a}$ is $|\lambda|$ times the magnitude of the vector $\boldsymbol{a}$, i.e.,

$$|\lambda\boldsymbol{a}| = |\lambda|\,|\boldsymbol{a}| \qquad \text{(see definition 4.4.2)}$$

A geometric visualisation of multiplication of a vector by a scalar is given in Fig. 4.13

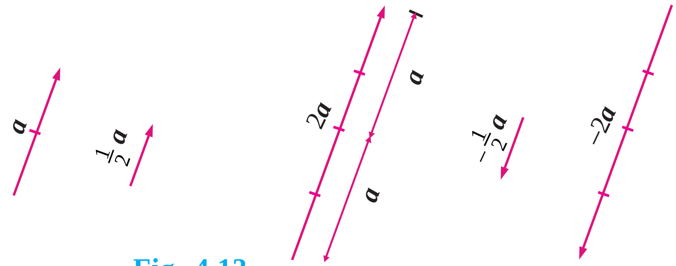


Fig. 4.13

When $\lambda = -1$, then $\lambda \boldsymbol{a} = -\boldsymbol{a}$, which is a vector having magnitude equal to the magnitude of $\boldsymbol{a}$ and direction opposite to that of the direction of $\boldsymbol{a}$. The vector $-\boldsymbol{a}$ is called the *negative* (or *additive inverse*) *of vector* $\boldsymbol{a}$, we always have

$$\boldsymbol{a} + (-\boldsymbol{a}) = (-\boldsymbol{a}) + \boldsymbol{a} = \mathbf{0}.$$

Also, if $\lambda = \dfrac{1}{|\boldsymbol{a}|}$, provided $\boldsymbol{a} \neq \mathbf{0}$ (i.e., $\boldsymbol{a}$ is not a null vector), then

$$|\lambda \boldsymbol{a}| = |\lambda||\boldsymbol{a}| = \frac{1}{|\boldsymbol{a}|}|\boldsymbol{a}| = 1.$$

So, $\lambda \boldsymbol{a}$ represents the unit vector $\hat{\boldsymbol{a}}$ in the direction of $\boldsymbol{a}$. Hence $\hat{\boldsymbol{a}} = \dfrac{1}{|\boldsymbol{a}|}\boldsymbol{a}$.

4.4.2 Definition

Let $\boldsymbol{a}$ be a vector and $\boldsymbol{\lambda}$ be a scalar. Then we define vector $\lambda \boldsymbol{a}$ to be the vector $\mathbf{0}$ if either $\boldsymbol{a}$ is the zero vector or $\boldsymbol{\lambda}$ is the zero scalar; otherwise $\lambda \boldsymbol{a}$ is the vector in the direction of $\boldsymbol{a}$ with magnitude $\lambda|\boldsymbol{a}|$ if $\lambda > 0$, and $\lambda \boldsymbol{a} = (-\lambda)(-\boldsymbol{a})$, if $\lambda < 0$.

Note : *If $\boldsymbol{\lambda} < 0$, then $\lambda \boldsymbol{a}$ is the vector in the opposite direction of $\boldsymbol{a}$ with magnitude $(-\lambda)\ |\boldsymbol{a}|$.*

4.4.3 Some laws of scalar multiplication of vector

We now state some laws of scalar multiplication of a vector which are useful for further discussion.

1. If $\boldsymbol{a}$ is a vector and λ is a scalar, then $(-\lambda)\boldsymbol{a} = \lambda(-\boldsymbol{a}) = -(\lambda \boldsymbol{a})$.

2. If $\boldsymbol{a}$ is a vector and m, n are scalars, then $m(n\boldsymbol{a}) = (mn)\boldsymbol{a} = (nm)\boldsymbol{a} = n(m\boldsymbol{a})$.

   In particular, if $n = -1$, then $m(-\boldsymbol{a}) = (-m)\boldsymbol{a} = -(m\boldsymbol{a})$.

3. If $\boldsymbol{a}$ is a vector and m, n are scalars, then $(m + n)\boldsymbol{a} = m\boldsymbol{a} + n\boldsymbol{a}$.

4.4.4 Theorem: *If m is a scalar and $\boldsymbol{a}, \boldsymbol{b}$ are any two vectors, then*

$$m(\boldsymbol{a} + \boldsymbol{b}) = m\boldsymbol{a} + m\boldsymbol{b}.$$

Proof: If $m = 0$ or one of $\boldsymbol{a}, \boldsymbol{b}$ is $\mathbf{0}$, then equality holds clearly.

Assume that $m \neq 0, \boldsymbol{a} \neq \mathbf{0}, \boldsymbol{b} \neq \mathbf{0}$

Case 1: $m > 0$.

Let $\mathbf{OA} = \boldsymbol{a}$, $\mathbf{AB} = \boldsymbol{b}$, $\mathbf{OA_1} = m\boldsymbol{a}$. Through A_1, draw parallel to $\boldsymbol{b}$ meeting the line OB in B_1.

Then $\mathbf{A_1B_1} = m\boldsymbol{b}$.

Then $\mathbf{OB} = \mathbf{OA} + \mathbf{AB} = \boldsymbol{a} + \boldsymbol{b}$. ... (1)

Since $m > 0$, $m(\boldsymbol{a}+\boldsymbol{b})$ and $\boldsymbol{a}+\boldsymbol{b}$ have the same direction.

Since ΔOAB and ΔOA_1B_1 are similar (Fig. 4.14)

$$\frac{OB_1}{OB} = \frac{A_1B_1}{AB} = m$$

$\therefore$ $\mathbf{A_1B_1} = m\mathbf{AB} = m\boldsymbol{b}$ and

$\mathbf{OB_1} = m\,\mathbf{OB}$

$\mathbf{OB_1} = \mathbf{OA_1} + \mathbf{A_1B_1} = m\boldsymbol{a} + m\boldsymbol{b}$ ... (2)

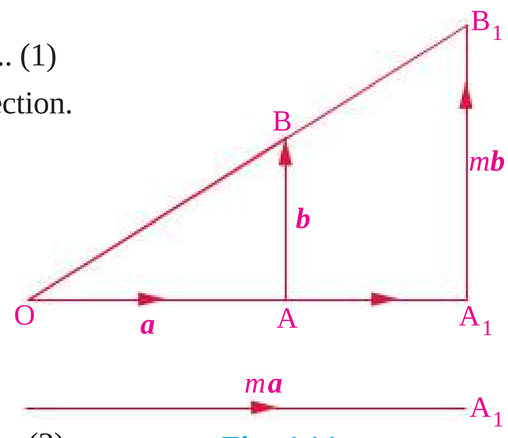


Fig. 4.14

By (1) and (2), $m\boldsymbol{a} + m\boldsymbol{b} = m(\boldsymbol{a}+\boldsymbol{b})$.

Case 2: $m < 0$ then $-m > 0$

$\therefore$ $m(\boldsymbol{a}+\boldsymbol{b}) = (-m)(-(\boldsymbol{a}+\boldsymbol{b}))$ (by definition)

$= (-m)(-\boldsymbol{a}-\boldsymbol{b})$

$= (-m)(-\boldsymbol{a}) + (-m)(-\boldsymbol{b})$ (by case 1)

$= m\boldsymbol{a} + m\boldsymbol{b}$ (by definition).

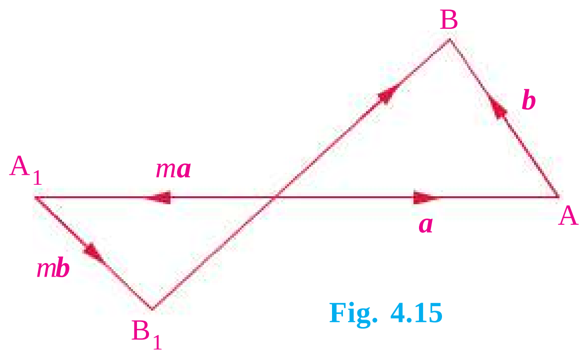


Fig. 4.15

4.4.5 Note

(i) *Two vectors are collinear (parallel) if and only if one is a scalar multiple of the other.*

(ii) *Three points* A, B *and* C *are collinear if and only if* **AB, BC** *are collinear vectors.*

4.5 Angle between two non-zero vectors

We have learnt about angles between two lines in plane geometry. We now introduce the concept of the angle between two non-zero vectors, which is slightly different from the angle between two lines. The concept of angle between two vectors is largely useful in Chapter 5, which deals with dot and cross products of two vectors.

4.5.1 Definition

Let $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be two non-zero vectors. Let* O, A *and* B *be points such that* $\mathbf{OA} = \boldsymbol{a}$ *and* $\mathbf{OB} = \boldsymbol{b}$. *Then the measure of* $\angle AOB$ *which lies between* 0^0 *and* 180^0 *is called the angle between* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *and is denoted by* $(\boldsymbol{a}, \boldsymbol{b})$ (see Fig. 4.16(a), (b), (c), (d)).

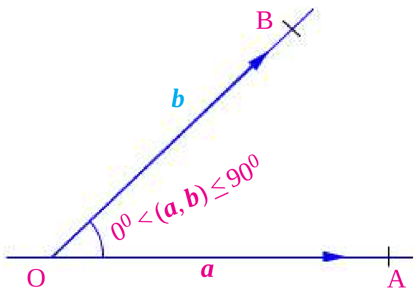


Fig. 4.16(a)

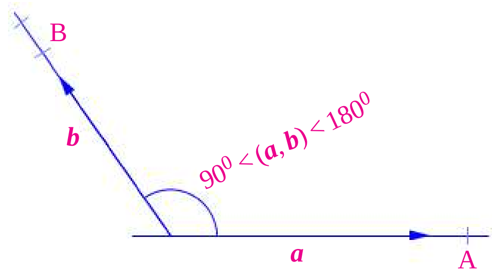


Fig. 4.16(b)

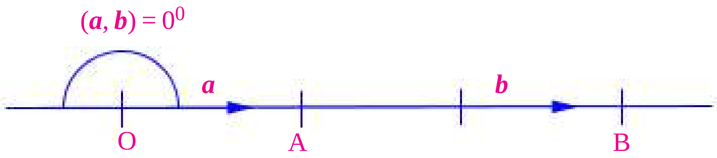


Fig. 4.16(c)

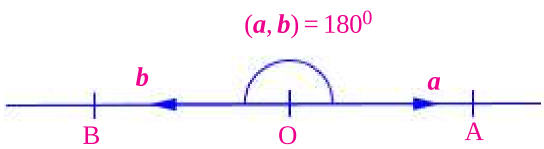


Fig. 4.16(d)

4.5.2 Note: *Let* $\mathbf{a}, \mathbf{b}$ *be non-zero vectors. Then*

(i) $(\mathbf{a}, \mathbf{b}) = 0^0 \Leftrightarrow \mathbf{a}$ *and* $\mathbf{b}$ *are like vectors.*

(ii) $(\mathbf{a}, \mathbf{b}) = 180^0 \Leftrightarrow \mathbf{a}$ *and* $\mathbf{b}$ *are unlike vectors.*

(iii) $(\mathbf{a}, \mathbf{b}) = 0^0$ *or* $180^0 \Leftrightarrow \mathbf{a}$ *and* $\mathbf{b}$ *are collinear vectors.*

(iv) *If* $(\mathbf{a}, \mathbf{b}) = 90^0$, *then* $\mathbf{a}, \mathbf{b}$ *are called perpendicular vectors.*

4.5.3 Note: *Let* $\mathbf{a}, \mathbf{b}$ *be non-zero vectors and* m, n *be positive scalars. Then*

(i) $(\mathbf{a}, \mathbf{b}) = (\mathbf{b}, \mathbf{a})$

(ii) $(\mathbf{a}, \mathbf{b}) = (-\mathbf{a}, -\mathbf{b})$

(iii) $(\mathbf{a}, -\mathbf{b}) = (-\mathbf{a}, \mathbf{b}) = 180^0 - (\mathbf{a}, \mathbf{b})$

(iv) $(\mathbf{a}, \mathbf{b}) = (m\,\mathbf{a},\ n\,\mathbf{b})$

(v) $(-m\,\mathbf{a}, n\,\mathbf{b}) = (m\,\mathbf{a}, -n\,\mathbf{b}) = 180^0 - (\mathbf{a}, \mathbf{b})$

Check these, by drawing the necessary diagrams.

4.5.4 Definition

Let A and B be two points and P, a point on the straight line AB. We say that P divides the line segment AB in the ratio $m : n\ (m + n \neq 0)$, *if* $n\,\mathbf{AP} = m\,\mathbf{PB}$.

4.5.5 Theorem: *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be position vectors of the points A and B with respect to the origin O. If a point P divides the line segment AB in the ratio* $m : n\ (m + n \neq 0)$, *then the position vector of P is* $\dfrac{m\mathbf{b} + n\mathbf{a}}{m + n}$. (if $k \neq 0$, then $\mathbf{a}/k$ or $\dfrac{\mathbf{a}}{k}$ means $\dfrac{1}{k}\,\mathbf{a}$)

Proof: Let P be the point on AB lying between A and B, in which case, P is said to divide AB internally. Let $\mathbf{OP} = \mathbf{r}$. By definition $n\,\mathbf{AP} = m\,\mathbf{PB}$.

$\Rightarrow \quad n\,(\mathbf{AO} + \mathbf{OP}) = m\,(\mathbf{PO} + \mathbf{OB})$

$\Rightarrow \quad n\,(\mathbf{OP} - \mathbf{OA}) = m\,(\mathbf{OB} - \mathbf{OP})$

$\Rightarrow \quad n\,(\boldsymbol{r} - \boldsymbol{a}) = m\,(\boldsymbol{b} - \boldsymbol{r})$

$\therefore \quad (m+n)\,\mathbf{r} = m\,\boldsymbol{b} + n\,\boldsymbol{a}.$

$$\therefore \quad \boldsymbol{r} = \frac{m\boldsymbol{b} + n\boldsymbol{a}}{m+n}$$

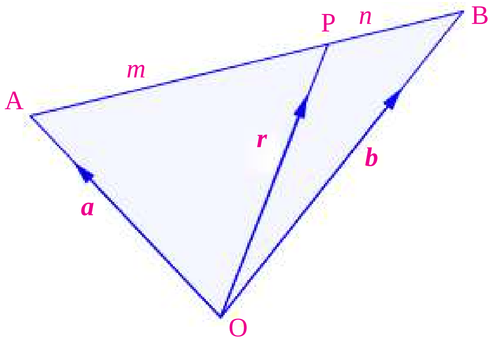


Fig. 4.17

Conversely if P is such that $\boldsymbol{r} = \mathbf{OP} = (m\boldsymbol{b} + n\mathbf{a}) / (m+n)$, then by retracing the above steps backwards, we can see that P lies on the line AB and $n\mathbf{AP} = m\mathbf{PB}$. Hence P divides AB in the ratio $m : n$.

4.5.6 Note: *The above formula is called (division)* ***section formula*** *and it holds whether* P *divides* AB *internally or externally. The position vector of the point P which divides the line segment AB externally (i.e.,* P *lies on* AB *outside the segment* AB*) in the ratio m : n is given by* $\boldsymbol{r} = (m\boldsymbol{b} - n\mathbf{a}) / (m-n)$.

4.5.7 Corollary: *If* P *is the mid point of* AB *then m = n and hence the position vector of* P = $\boldsymbol{r} = \mathbf{OP} = (\boldsymbol{a} + \boldsymbol{b}) / 2$.

Proof: In Theorem 4.5.5, take $m = n = 1$.

4.5.8 Theorem: *Let* $\boldsymbol{a}, \boldsymbol{b}$ *be any two non-collinear vectors. If* $\boldsymbol{r}$ *is any vector in the plane* Π *determined by a pair of supports of* $\boldsymbol{a}$ *and* $\boldsymbol{b}$*, then there exist unique scalars x and y such that*

$$\boldsymbol{r} = x\,\boldsymbol{a} + y\,\boldsymbol{b}$$

Proof: Choose a point 'O' in the plane Π as the origin and points A and B in Π. $\boldsymbol{a} = \mathbf{OA}$ and $\boldsymbol{b} = \mathbf{OB}$ so that O, A and B are not collinear.

Let P be a point in the plane Π such that $\mathbf{OP} = \boldsymbol{r}$. If P lies either on the support of $\boldsymbol{a}$ (i.e., on the line OA) or on the support of $\boldsymbol{b}$ (i.e. the line OB), then take $y = 0$ or $x = 0$ respectively.

Suppose P does not lie on the supports of $\boldsymbol{a}$ and $\boldsymbol{b}$. Through P draw lines parallel to $\boldsymbol{b}$ meeting the support of $\boldsymbol{a}$ in L and parallel to $\boldsymbol{a}$ meeting the support of $\boldsymbol{b}$ in M. Thus **OL** is collinear with $\boldsymbol{a}$ and **OM** is collinear with $\boldsymbol{b}$ (see Fig. 4.18).

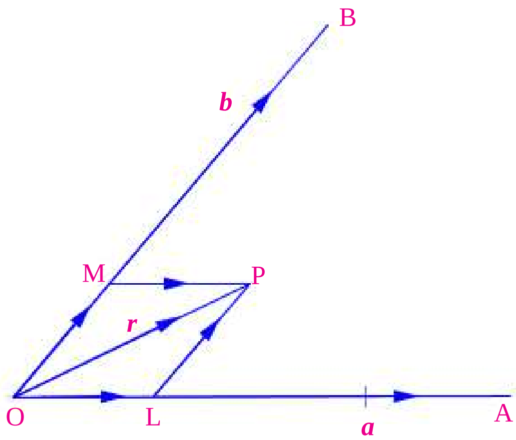


Fig. 4.18

Hence there exist scalars x and y such that $\mathbf{OL} = x\,\boldsymbol{a}$ and $\mathbf{OM} = y\,\boldsymbol{b}$.

Then $\boldsymbol{r} = \mathbf{OP} = \mathbf{OL} + \mathbf{LP}$

$= \mathbf{OL} + \mathbf{OM} = x\,\boldsymbol{a} + y\,\boldsymbol{b}$.

If $\boldsymbol{r}$ is also equal to $x'\boldsymbol{a} + y'\boldsymbol{b}$, then $(x - x')\,\boldsymbol{a} = (y' - y)\,\boldsymbol{b}$ so that $x = x'$, $y = y'$, otherwise $\boldsymbol{a}$ and $\boldsymbol{b}$ will be collinear vectors. Thus x and y are unique.

4.5.9 Corollary: *If $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-collinear vectors and x, y are scalars, then $x\,\boldsymbol{a} + y\,\boldsymbol{b} = \mathbf{0}$ if and only if $x = y = 0$.*

Proof: If $x = y = 0$, then $x\,\boldsymbol{a} + y\,\boldsymbol{b} = \mathbf{0}$. Suppose that $x\,\boldsymbol{a} + y\,\boldsymbol{b} = \mathbf{0}$.

Since $\mathbf{0} = 0\,\boldsymbol{a} + 0\,\boldsymbol{b}$, by Theorem 4.5.8 , $x = 0 = y$.

It is known that non–coplanar vectors do exist in the space and in particular three non–coplanar vectors with the same initial point exist. Now, we have the following theorem which we call as **space representation theorem**.

4.5.10 Theorem: *Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ be three non-coplanar vectors and $\boldsymbol{r}$ be any vector in the space. Then, there exists unique triad of scalars x, y, z such that*

$$\boldsymbol{r} = x\,\boldsymbol{a} + y\,\boldsymbol{b} + z\,\boldsymbol{c}$$

Proof: Let 'O' be the origin, $\mathbf{OA} = \boldsymbol{a}$, $\mathbf{OB} = \boldsymbol{b}$ and $\mathbf{OC} = \boldsymbol{c}$. Let P be a point in the space and $\boldsymbol{r} = \mathbf{OP}$.

If P lies on the support of $\boldsymbol{a}$ that is, $\boldsymbol{r}$ is collinear with $\boldsymbol{a}$, then we choose $y = 0 = z$.

Similarly, if P lies on the support of $\boldsymbol{b}$ or $\boldsymbol{c}$, then choose $z = 0 = x$ or $x = 0 = y$ respectively.

Suppose P lies in the plane of **OA** and **OB**. Then by Theorem 4.5.8, $\boldsymbol{r} = x\boldsymbol{a} + y\boldsymbol{b}$ so that $z = 0$. Similarly if P lies in the plane of **OB** and **OC**, then $\boldsymbol{r} = y\,\boldsymbol{b} + z\,\boldsymbol{c}$, $x = 0$ and, if P lies in the plane of **OC** and **OA**, then $\boldsymbol{r} = x\,\boldsymbol{a} + z\,\boldsymbol{c}$, $y = 0$.

Now suppose P does not belong to any of the planes BOC, COA and AOB. Through the point P draw planes parallel to the planes BOC, COA and AOB meeting the supports of $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ in L, M and N respectively (see Fig. 4.19).

Thus we form the space figure P Q L R M S N O which is called a parallelopiped.

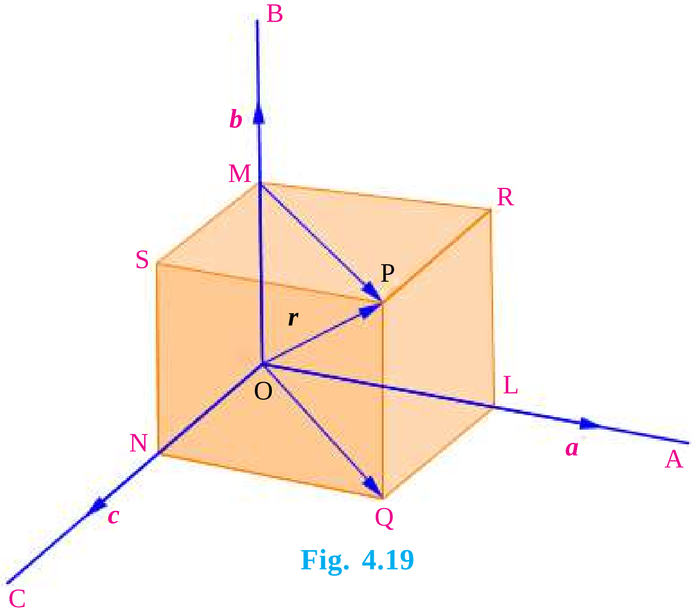


Fig. 4.19

Now, $\boldsymbol{r} = \mathbf{OP} = \mathbf{OQ} + \mathbf{QP} = (\mathbf{OL} + \mathbf{LQ}) + \mathbf{OM} = (\mathbf{OL} + \mathbf{ON}) + \mathbf{OM}$

$= \mathbf{OL} + \mathbf{OM} + \mathbf{ON}$.

Since **OL**, **OM** and **ON** are collinear with ***a***, ***b*** and **c** respectively, there exist scalars x, y and z such that **OL** = $x\,\boldsymbol{a}$, **OM** = $y\,\boldsymbol{b}$ and **ON** = $z\,\boldsymbol{c}$.

$$\therefore\ \boldsymbol{r} = x\boldsymbol{a} + y\boldsymbol{b} + z\boldsymbol{c}.$$

If $\boldsymbol{r}$ is also equal to $x'\boldsymbol{a} + y'\boldsymbol{b} + z'\boldsymbol{c}$, then $(y - y')\boldsymbol{b} + (z - z')\boldsymbol{c} = (x' - x)\boldsymbol{a}$.

If $x \neq x'$, then $\boldsymbol{a}$ is coplanar with $\boldsymbol{b}$ and $\boldsymbol{c}$ (Theorem 4.5.8) which is not true.

$$\therefore\ x = x'.\ \text{Similarly } y = y' \text{ and } z = z'.$$

4.5.11 Corollary : If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are non–coplanar vectors, then $x\boldsymbol{a} + y\boldsymbol{b} + z\boldsymbol{c} = \boldsymbol{0}$ if and only if $x = y = z = 0$.

Proof: If $x = y = z = 0$, then clearly $x\boldsymbol{a} + y\boldsymbol{b} + z\boldsymbol{c} = \boldsymbol{0}$.

Suppose $x\boldsymbol{a} + y\boldsymbol{b} + z\boldsymbol{c} = \boldsymbol{0}$. Since $\boldsymbol{0} = 0\,\boldsymbol{a} + 0\,\boldsymbol{b} + 0\,\boldsymbol{c}$ by Theorem 4.5.10, $x = 0$, $y = 0$, $z = 0$.

4.6 Linear Combination of Vectors

This section is devoted to discuss the linear combinations of vectors.

4.6.1 Definition

*Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \ldots, \boldsymbol{a}_n$ be vectors and $x_1, x_2, x_3, \ldots, x_n$ be scalars. Then the vector $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3 + \ldots + x_n\boldsymbol{a}_n$ is called a **linear combination** of the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \ldots, \boldsymbol{a}_n$.*

4.6.2 Note

(i) $2\boldsymbol{a} - \boldsymbol{b} + 3\boldsymbol{c}$ is a linear combination of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.

(ii) If ***a***, ***b*** are non-collinear vectors, then by Theorem 4.5.8, every vector in the plane determined by a pair of supports of ***a*** and ***b*** can be expressed as linear combination of ***a*** and ***b*** in one and only one way.

(iii) If ***a***, ***b***, ***c*** be three non-coplanar vectors, then Theorem 4.5.10 shows that every vector in the space can be expressed as a linear combination of ***a***, ***b***, ***c*** in one and only one way.

(iv) Three vectors are coplanar if and only if one of them is a linear combination of the other two.

4.7 Components of a vector in Three Dimensions

In Theorem 4.5.10 we have proved that every vector can be expressed as a linear combination of three non-coplanar vectors. Here we introduce the concept of components of a vector with respect to given non coplanar vectors ***a***, ***b***, ***c***.

4.7.1 Definition (Components)

*Consider the ordered triad (**a, b, c**) of non-coplanar vectors **a, b, c**. If **r** is any vector then it is proved in Theorem 4.5.10 that there exists unique triad (x, y, z) of scalars such that **r** = x**a** + y**b** + z**c**. These scalars x, y, z are called the **components** of **r** with respect to the ordered triad (**a, b, c**).*

Any ordered triad of non-coplanar vectors is called a base for the space.

The components of a vector depend on the choice of the base.

4.7.2 Representing a vector in component form

We shall now express a given vector in component form. Let O be a point in space. Call it the origin. Take three mutually perpendicular X, Y and Z axes. Let us take the points A(1, 0, 0), B(0, 1, 0) and C(0, 0, 1) on the X-axis, Y-axis and Z-axis, respectively. Then clearly

|**OA**| = 1, |**OB**| = 1 and |**OC**| = 1.

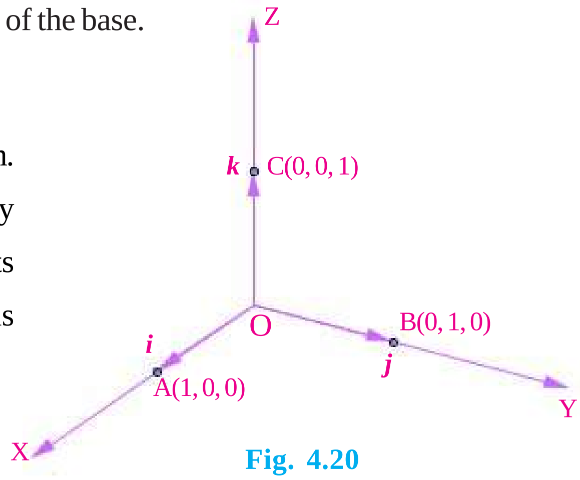


Fig. 4.20

The vectors **OA**, **OB** and **OC**, each having magnitude 1, are called unit vectors along the axes OX, OY and OZ, respectively, and denoted by ***i***, ***j*** and ***k***, respectively (Fig. 4.20).

Now, consider the position vector **OP** of a point P(x, y, z) as in Fig. 4.21. Let P_1 be the foot of the perpendicular from P on the plane XOY. We thus see that P_1P is parallel to Z-axis.

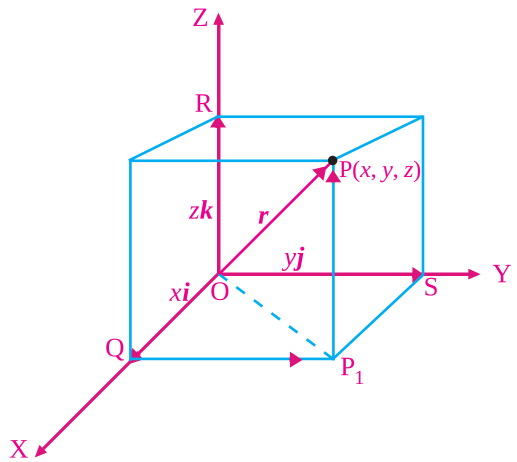


Fig. 4.21

As ***i***, ***j*** and ***k*** are the unit vectors along the X, Y and Z-axes, respectively, and by the definition of the coordinates of P, we have $\mathbf{P_1P} = \mathbf{OR} = z\boldsymbol{k}$.

Similarly $\mathbf{QP}_1 = \mathbf{OS} = y\boldsymbol{j}$ and $\mathbf{OQ} = x\boldsymbol{i}$.

Therefore, it follows that $\mathbf{OP}_1 = \mathbf{OQ} + \mathbf{QP}_1 = x\boldsymbol{i} + y\boldsymbol{j}$

and $\mathbf{OP} = \mathbf{OP}_1 + \mathbf{P_1P} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.

Hence, the position vector of P with reference to O is given by

$$\mathbf{OP} \text{ (or } \boldsymbol{r}) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}.$$

This form of any vector is called its ***component form***. Here, x, y and z are called the ***scalar components*** of $\boldsymbol{r}$, and $x\boldsymbol{i}$, $y\boldsymbol{j}$ and $z\boldsymbol{k}$ are called the ***vector components*** of $\boldsymbol{r}$ along the respective axes. Sometimes x, y and z are also termed as ***rectangular components***.

4.7.3 Length of a vector in terms of its components

The length of any vector $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, is readily determined by applying the Pythagoras theorem twice. We note that in the right angle triangle OQP_1 (Fig. 4.21)

$$|\mathbf{OP}_1| = \sqrt{|\mathbf{OQ}|^2 + |\mathbf{QP}_1|^2} = \sqrt{x^2 + y^2},$$

and in the right angle triangle OP_1P, we have

$$|\mathbf{OP}_1| = \sqrt{|\mathbf{OP}_1|^2 + |\mathbf{P_1P}|^2} = \sqrt{(x^2 + y^2) + z^2}.$$

Hence, the length of any vector $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ is given by

$$|\boldsymbol{r}| = |x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}| = \sqrt{x^2 + y^2 + z^2}.$$

4.7.4 Note : If $\boldsymbol{a}$ and $\boldsymbol{b}$ are any two vectors given in the component form $a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ respectively, then the following results of addition, subtraction and scalar multiplication of vectors hold in component form :

(i) the sum (or resultant) of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by

$$\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1)\boldsymbol{i} + (a_2 + b_2)\boldsymbol{j} + (a_3 + b_3)\boldsymbol{k}$$

(ii) the difference of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by

$$\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1)\boldsymbol{i} + (a_2 - b_2)\boldsymbol{j} + (a_3 - b_3)\boldsymbol{k}$$

(iii) the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are equal if and only if $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.

(iv) the multiplication of vector $\boldsymbol{a}$ by any scalar λ is given by

$$\lambda\boldsymbol{a} = (\lambda a_1)\boldsymbol{i} + (\lambda a_2)\boldsymbol{j} + (\lambda a_3)\boldsymbol{k}.$$

4.7.5 Vector joining two points

If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are any two points, then the vector joining P_1 and P_2 is the vector $\mathbf{P_1P_2}$ (Fig. 4.22).

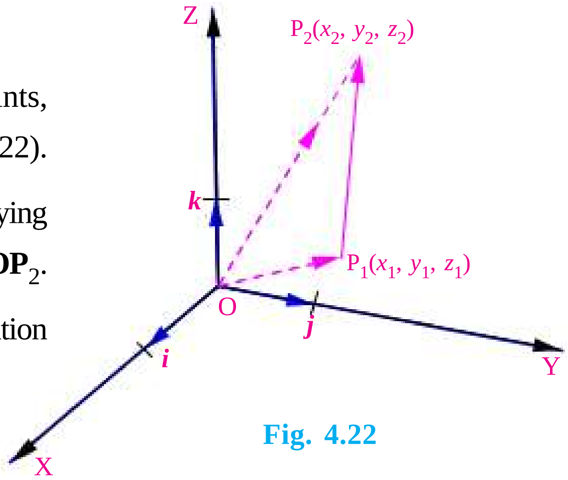


Fig. 4.22

Joining the points P_1 and P_2 with the origin O, and applying triangle law, to the triangle OP_1P_2, we have $\mathbf{OP_1} + \mathbf{P_1P_2} = \mathbf{OP_2}$.

Using the properties of vector addition, the above equation becomes

$$\mathbf{P_1P_2} = \mathbf{OP_2} - \mathbf{OP_1}$$

i.e.,
$$\mathbf{P_1P_2} = (x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}) - (x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k})$$
$$= (x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j} + (z_2 - z_1)\boldsymbol{k}$$

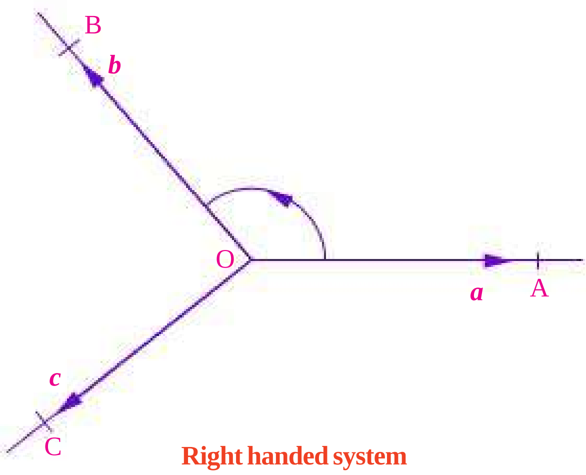
The magnitude of vector $\mathbf{P_1P_2}$ is given by

$$|\mathbf{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

4.7.6 Definition (Right handed and left handed triads)

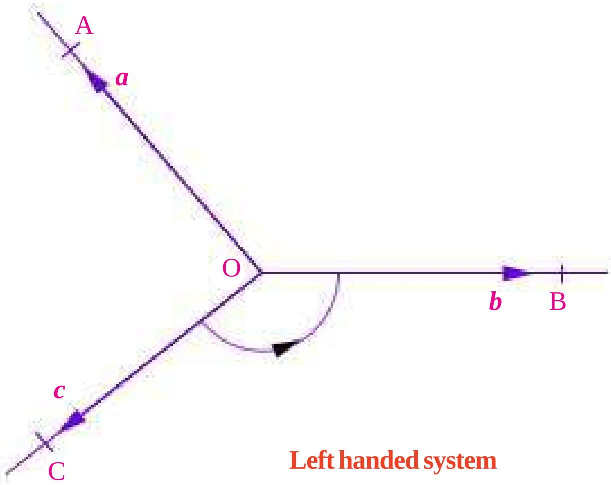
Let $\mathbf{OA} = \boldsymbol{a}$, $\mathbf{OB} = \boldsymbol{b}$, $\mathbf{OC} = \boldsymbol{c}$ be three non-coplanar vectors.

Viewing from the point C, if the rotation of **OA** to **OB** does not exceed angle 180^0 in anti-clock sense, then $\boldsymbol{a, b, c}$ are said to form a **right handed system of vectors** and we say simply that $(\boldsymbol{a, b, c})$ is a right handed system. If $(\boldsymbol{a, b, c})$ is not a right handed system, then it is called a **left handed system** (see Fig. 4.23(a) and 4.23(b)).



Right handed system

Fig. 4.23(a)



Left handed system

Fig. 4.23(b)

4.7.7 Solved Problems

1. Problem: *Find unit vector in the direction of vector $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$.*

Solution: The unit vector in the direction of a vector $\boldsymbol{a}$ is given by $\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$.

Now $|\boldsymbol{a}| = \sqrt{2^2+3^2+1^2} = \sqrt{14}$.

Therefore $\hat{\boldsymbol{a}} = \frac{1}{\sqrt{14}}(2\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}) = \frac{2}{\sqrt{14}}\boldsymbol{i}+\frac{3}{\sqrt{14}}\boldsymbol{j}+\frac{1}{\sqrt{14}}\boldsymbol{k}.$

2. Problem: *Find a vector in the direction of vector $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j}$ that has magnitude 7 units.*

Solution: The unit vector in the direction of the given vector $\boldsymbol{a}$ is

$$\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a} = \frac{1}{\sqrt{5}}(\boldsymbol{i}-2\boldsymbol{j}) = \frac{1}{\sqrt{5}}\boldsymbol{i}-\frac{2}{\sqrt{5}}\boldsymbol{j}.$$

Therefore, the vector having magnitude equal to 7 and in the direction of $\boldsymbol{a}$ is

$$7\boldsymbol{a} = 7\left(\frac{1}{\sqrt{5}}\boldsymbol{i}-\frac{2}{\sqrt{5}}\boldsymbol{j}\right) = \frac{7}{\sqrt{5}}\boldsymbol{i}-\frac{14}{\sqrt{5}}\boldsymbol{j}.$$

3. Problem: *Find the unit vector in the direction of the sum of the vectors,*

$\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j} - 5\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$.

Solution: The sum of the given vectors is $\boldsymbol{a}+\boldsymbol{b}\ (=\boldsymbol{c}, \text{say}) = 4\boldsymbol{i}+3\boldsymbol{j}-2\boldsymbol{k}$

and $|\boldsymbol{c}| = \sqrt{4^2+3^2+(-2)^2} = \sqrt{29}$. $\therefore\ \hat{\boldsymbol{c}} = \frac{\boldsymbol{a}+\boldsymbol{b}}{|\boldsymbol{a}+\boldsymbol{b}|} = \frac{4\boldsymbol{i}+3\boldsymbol{j}-2\boldsymbol{k}}{\sqrt{29}}$.

4. Problem: *Write direction ratios of the vector $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$ and hence calculate its direction cosines.*

Solution: Note that direction ratios a, b, c of a vector $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ are just the respective components x, y and z of the vector. So, for the given vector, we have $a = 1, b = 1, c = -2$. Further, if l, m and n the direction cosines of the given vector, then

$$l = \frac{a}{|\boldsymbol{r}|} = \frac{1}{\sqrt{6}},\ m = \frac{b}{|\boldsymbol{r}|} = \frac{1}{\sqrt{6}},\ n = \frac{c}{|\boldsymbol{r}|} = -\frac{2}{\sqrt{6}} \text{ as } |\boldsymbol{r}| = \sqrt{6}.$$

Thus, the direction cosines are $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$.

5. Problem: *Consider two points P and Q with position vectors* **OP** *$= 3\boldsymbol{a} - 2\boldsymbol{b}$ and* **OQ** *$= \boldsymbol{a} + \boldsymbol{b}$. Find the position vector of a point R which divides the line joining P and Q in the ratio 2 : 1, (i) internally and (ii) externally.*

Solution:

(i) The position vector of the point R dividing the join of P and Q internally in the ratio 2 : 1 is

$$\mathbf{OR} = \frac{2(\boldsymbol{a}+\boldsymbol{b})+(3\boldsymbol{a}-2\boldsymbol{b})}{2+1} = \frac{5\boldsymbol{a}}{3}.$$

(ii) The position vector of the point R dividing the join of P and Q externally in the ratio 2 :1 is

$$\mathbf{OR} = \frac{2(\boldsymbol{a}+\boldsymbol{b})-(3\boldsymbol{a}-2\boldsymbol{b})}{2-1} = 4\boldsymbol{b}-\boldsymbol{a}.$$

6. Problem: *Show that the points A(2**i** – **j** + **k**), B(**i** – 3**j** – 5**k**), C(3**i** – 4**j** – 4**k**) are the vertices of a right angled triangle.*

Solution: We have

$\mathbf{AB} = (1-2)\boldsymbol{i} + (-3+1)\boldsymbol{j} + (-5-1)\boldsymbol{k} = -\boldsymbol{i} - 2\boldsymbol{j} - 6\boldsymbol{k}.$

$\mathbf{BC} = (3-1)\boldsymbol{i} + (-4+3)\boldsymbol{j} + (-4+5)\boldsymbol{k} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}.$

and $\mathbf{CA} = (2-3)\boldsymbol{i} + (-1+4)\boldsymbol{j} + (1+4)\boldsymbol{k} = -\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}.$

we have $|\mathbf{AB}|^2 = |\mathbf{BC}|^2 + |\mathbf{CA}|^2.$

7. Problem: *Let* **A, B, C** *and* **D** *be four points with position vectors* $\boldsymbol{a} + 2\boldsymbol{b}$, $2\boldsymbol{a} - \boldsymbol{b}$, $\boldsymbol{a}$ *and* $3\boldsymbol{a} + \boldsymbol{b}$ *respectively. Express the vectors* **AC, DA, BA** *and* **BC** *interms of* ***a*** *and* ***b*.**

Solution: Let 'O' be the origin of reference so that $\mathbf{OA} = \boldsymbol{a} + 2\boldsymbol{b}$, $\mathbf{OB} = 2\boldsymbol{a} - \boldsymbol{b}$, $\mathbf{OC} = \boldsymbol{a}$ and $\mathbf{OD} = 3\boldsymbol{a} + \boldsymbol{b}$. Then

$\mathbf{AC} = \mathbf{OC} - \mathbf{OA}$

$= \boldsymbol{a} - (\boldsymbol{a} + 2\boldsymbol{b}) = -2\boldsymbol{b}$

$\mathbf{DA} = (\boldsymbol{a} + 2\boldsymbol{b}) - (3\boldsymbol{a} + \boldsymbol{b}) = -2\boldsymbol{a} + \boldsymbol{b}$

$\mathbf{BA} = (\boldsymbol{a} + 2\boldsymbol{b}) - (2\boldsymbol{a} - \boldsymbol{b}) = 3\boldsymbol{b} - \boldsymbol{a}$

$\mathbf{BC} = \boldsymbol{a} - (2\boldsymbol{a} - \boldsymbol{b}) = \boldsymbol{b} - \boldsymbol{a}.$

8. Problem: Let A B C D E F *be a regular hexagon with centre 'O'. Show that*

AB + **AC** + **AD** + **AE** + **AF** = 3 **AD** = 6 **AO**.

Solution: From Fig. 4.23

AB + **AC** + **AD** + **AE** + **AF**

= (**AB** + **AE**) + **AD** + (**AC** + **AF**)

= (**AE** + **ED**) + **AD** + (**AC** + **CD**) (Fig. 4.24)

(∵ **AB** = **ED**, **AF** = **CD**)

= **AD** + **AD** + **AD** = 3**AD**

= 6**AO** (∵ 'O' is the centre and **OD = AO**).

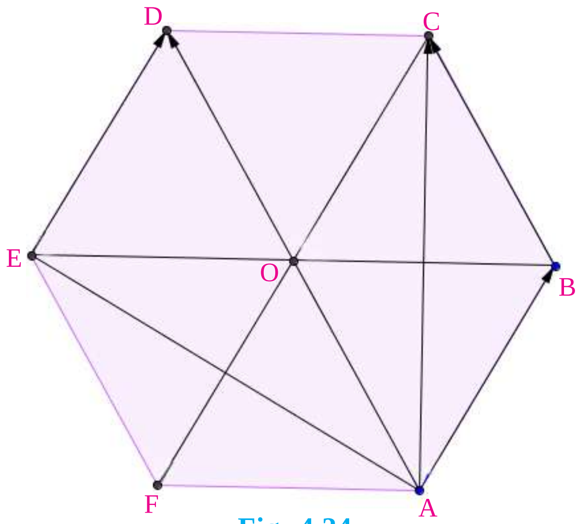


Fig. 4.24

9. Problem: *In* Δ ABC, *if* ***a***, ***b***, ***c*** *are position vectors of the vertices* A, B *and* C *respectively, then prove that the position vector of the centroid* G *is* $\frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$.

Solution: Let G be the centroid of Δ ABC and A D the median through the vertex A. (see Fig. 4.25)

Then AG : GD = 2 : 1.

Since the position vector of D is $\frac{1}{2}$ $(\boldsymbol{b} + \boldsymbol{c})$,

by the Theorem 4.5.5 , the position vector of G is

$$\frac{\frac{2\,(\boldsymbol{b}+\boldsymbol{c})}{2} + 1\boldsymbol{a}}{2+1} = \frac{\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}}{3}.$$

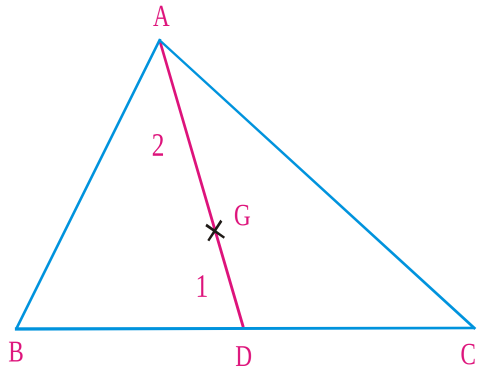


Fig. 4.25

10. Problem: *In* Δ ABC, *if* 'O' *is the circumcentre and* H *is the orthocentre, then show that*

(i) **OA** + **OB** + **OC** = **OH** (ii) **HA** + **HB** + **HC** = 2 **HO**

Solution: Let D be the mid point of BC.

(i) Take 'O' as the origin, let **OA** = ***a***; **OB** = ***b*** an **OC** = ***c*** (See Fig.4.26)

$$\text{OD} = \frac{\boldsymbol{b}+\boldsymbol{c}}{2}$$

∴ **OA**+**OB**+**OC** = **OA**+2**OD** = **OA**+**AH** = **OH**

(Observe that AH = 2R cos A, OD = R cos A,

R is the circum radius of Δ ABC and hence **A H** = 2 **OD**)

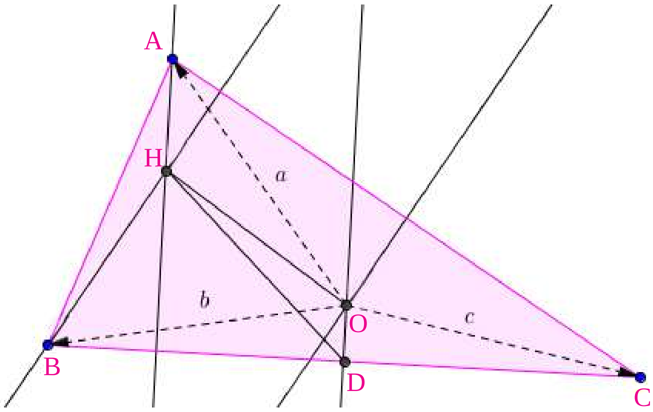


Fig. 4.26

(ii) **HA** + **HB** + **HC** = **HA** + 2**HD** = **HA** + 2(**HO** + **OD**)

= **HA** + 2**HO** + 2**OD** = **HA** + 2**HO** + **AH** = 2**HO**.

Note : *Taking circumcentre as the origin, we have proved that the position vector of the orthocentre of a triangle is the sum of the position vectors of the vertices which will be very useful in proving geometrical problems concerning triangles.*

11. Problem: *Let* ***a***, ***b***, ***c*** *and* ***d*** *be the position vectors of* A, B, C *and* D *respectively which are the vertices of a tetrahedron. Then prove that the lines joining the vertices to the centroids of the opposite faces are concurrent* (*this point is called the centroid or the centre of the tetrahedron*).

Solution : Let O be the origin of reference. Let G_1, G_2, G_3 and G_4 be the centroids of Δ BCD, Δ CAD, Δ ABD and Δ ABC respectively (see Fig.4.27).

$$\text{Then } \mathbf{OG_1} = \frac{\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}}{3}.$$

Consider the point P that divides AG_1 in the ratio 3 : 1.

$$OP = \frac{\frac{3(\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d})}{3} + 1\boldsymbol{a}}{4}$$

$$\therefore\ OP = \frac{1}{4}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}).$$

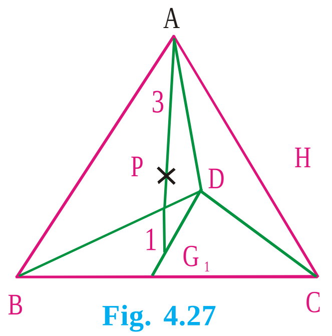


Fig. 4.27

Similarly we can show that the position vectors of the points dividing BG_2, CG_3 and DG_4 in the ratio 3 : 1 are equal to $\frac{1}{4}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d})$. Therefore P lies on each of AG_1, BG_2, CG_3 and DG_4.

12. Problem: *Let* OABC *be a parallelogram and* D *the midpoint of* OA. *Prove that the segment* CD *trisects the diagonal* OB *and is trisected by the diagonal* OB.

Solution: Let $\mathbf{OA} = \boldsymbol{a}$, $\mathbf{OC} = \boldsymbol{b}$ so that $\mathbf{OB} = \boldsymbol{a} + \boldsymbol{b}$; $\mathbf{OD} = \frac{\boldsymbol{a}}{2}$.

Let M be the point of intersection of OB and CD (see Fig. 4.28).
Let OM : MB = k : 1 and CM : MD = l : 1.

$$\therefore\ \mathbf{OM} = \frac{k(\boldsymbol{a}+\boldsymbol{b})}{k+1} \text{ and also}$$

$$\mathbf{OM} = \frac{l\left(\frac{\boldsymbol{a}}{2}\right) + 1\boldsymbol{b}}{l+1} = \frac{l\boldsymbol{a} + 2\boldsymbol{b}}{2(l+1)}$$

$$\therefore \frac{l}{2(l+1)} = \frac{k}{k+1} = \frac{1}{l+1}$$

$$\therefore\ l = 2 \text{ and } k = \frac{1}{2}.$$

$\therefore$ CD trisects OB and OB trisects CD.

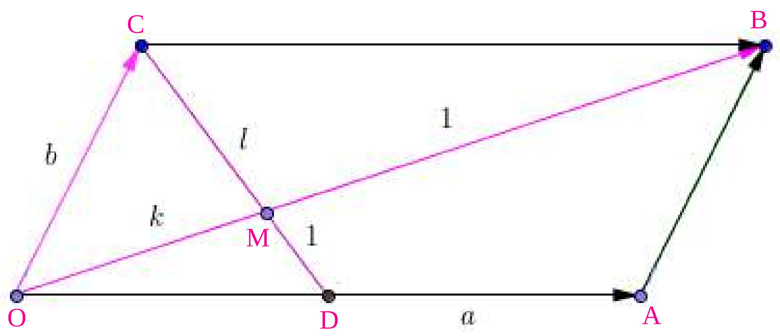


Fig. 4.28

13. Problem: *Let* $\boldsymbol{a}, \boldsymbol{b}$ *be non-collinear vectors. If* $\boldsymbol{\alpha} = (x + 4y)\,\boldsymbol{a} + (2x + y + 1)\,\boldsymbol{b}$ *and* $\boldsymbol{\beta} = (y - 2x + 2)\,\boldsymbol{a} + (2x - 3y - 1)\,\boldsymbol{b}$ *are such that* $3\boldsymbol{\alpha} = 2\boldsymbol{\beta}$, *then find* x *and* y.

Solution: $3\boldsymbol{\alpha} = 2\boldsymbol{\beta} \Rightarrow 3(x + 4y)\,\boldsymbol{a} + 3(2x + y + 1)\,\boldsymbol{b} = 2(y - 2x + 2)\,\boldsymbol{a} + 2(2x - 3y - 1)\boldsymbol{b}$

On comparing the coefficients of $\boldsymbol{a}$ and $\boldsymbol{b}$, we have

$$3x + 12y = 2y - 4x + 4 \quad \Rightarrow \quad 7x + 10y = 4 \qquad \ldots (1)$$

and $6x + 3y + 3 = 4x - 6y - 2 \Rightarrow 2x + 9y = -5$ ...(2)

Solving (1) and (2), $x = 2, y = -1$.

14. Problem: *Show that the points whose position vectors are* $-2\boldsymbol{a} + 3\boldsymbol{b} + 5\boldsymbol{c}$, $\boldsymbol{a} + 2\boldsymbol{b} + 3\boldsymbol{c}$, $7\boldsymbol{a} - \boldsymbol{c}$ *are collinear when* ***a, b, c*** *are non-coplanar vectors.*

Solution: Let P, Q, R be the given points.

Then $\mathbf{PQ} = 3\boldsymbol{a} - \boldsymbol{b} - 2\boldsymbol{c}$, $\mathbf{QR} = 6\boldsymbol{a} - 2\boldsymbol{b} - 4\boldsymbol{c}$

$\therefore$ $\mathbf{QR} = 2\,\mathbf{PQ}$. Hence P, Q and R are collinear.

15. Problem: *If the points whose position vectors are* $3\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$, $2\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}$, $-\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ *and* $4\boldsymbol{i} + 5\boldsymbol{j} + \lambda\boldsymbol{k}$ *are coplanar, then show that* $\lambda = -\dfrac{146}{17}$.

Solution: Let the given points be A, B, C and D respectively.

Then $\mathbf{AB} = -\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k}$, $\mathbf{AC} = -4\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}$ and

$\mathbf{AD} = \boldsymbol{i} + 7\boldsymbol{j} + (\lambda + 1)\boldsymbol{k}$.

A, B, C and D are coplanar if and only if

$\mathbf{AD} = x\,\mathbf{AB} + y\,\mathbf{AC}$, for some scalars x, y; that is

$\boldsymbol{i} + 7\boldsymbol{j} + (\lambda + 1)\boldsymbol{k} = x(-\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k}) + y(-4\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k})$

Equating the corresponding coefficients

$$-x - 4y = 1,\ 5x + 3y = 7,\ -3x + 3y = \lambda + 1$$

Solving the first two equations we get $x = \dfrac{31}{17}$, $y = -\dfrac{12}{17}$

and hence $\lambda = -3x + 3y - 1 = -\dfrac{146}{17}$.

Exercise 4(a)

I. 1. ABCD is a parallelogram. If L and M are the middle points of BC and CD respectively, then find

(i) **AL** and **AM** interms of **AB** and **AD**.

(ii) λ, if $\mathbf{AM} = \lambda\mathbf{AD} - \mathbf{LM}$

2. In ΔABC, P, Q and R are the midpoints of the sides AB, BC and CA respectively. If D is any point

(i) then express **DA + DB + DC** interms of **DP**, **DQ** and **DR**

(ii) if $\mathbf{PA} + \mathbf{QB} + \mathbf{RC} = \boldsymbol{\alpha}$, then find $\boldsymbol{\alpha}$.

3. Let $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{b} = 3\boldsymbol{i} + \boldsymbol{j}$. Find the unit vector in the direction of $\boldsymbol{a} + \boldsymbol{b}$.

4. If the vectors $-3\boldsymbol{i} + 4\boldsymbol{j} + \lambda\boldsymbol{k}$ and $\mu\boldsymbol{i} + 8\boldsymbol{j} + 6\boldsymbol{k}$ are collinear vectors, then find λ and μ.

5. ABCDE is a pentagon. If the sum of the vectors **AB**, **AE, BC**, **DC**, **ED** and **AC** is λ**AC,** then find the value of λ.

6. If the position vectors of the points A, B and C are $-2\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}$, $-4\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k}$ and $6\boldsymbol{i}-3\boldsymbol{j}-13\boldsymbol{k}$ respectively and **AB** = λ**AC**, then find the value of λ.

7. If **OA** $=\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$, **AB** $=3\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k}$, **BC** $=\boldsymbol{i}+2\boldsymbol{j}-2\boldsymbol{k}$ and **CD** $=2\boldsymbol{i}+\boldsymbol{j}+3\boldsymbol{k}$, then find the vector **OD**.

8. $\boldsymbol{a}=2\boldsymbol{i}+5\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{b}=4\boldsymbol{i}+m\boldsymbol{j}+n\boldsymbol{k}$ are collinear vectors, then find m and n.

9. Let $\boldsymbol{a}=2\boldsymbol{i}+4\boldsymbol{j}-5\boldsymbol{k}$, $\boldsymbol{b}=\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{c}=\boldsymbol{j}+2\boldsymbol{k}$. Find the unit vector in the opposite direction of $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$.

10. Is the triangle formed by the vectors $3i+5j+2k$, $2i-3j-5k$ and $-5i-2j+3k$ equilateral?

11. If α, β and γ are the angles made by the vector $3\boldsymbol{i}-6\boldsymbol{j}+2\boldsymbol{k}$ with the positive directions of the coordinate axes then find $\cos\alpha$, $\cos\beta$ and $\cos\gamma$.

12. Find the angles made by the straight line passing through the points $(1, -3, 2)$ and $(3, -5, 1)$ with the coordinate axes.

II. 1. If $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}=\alpha\boldsymbol{d}$, $\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}=\beta\boldsymbol{a}$ and $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are non-coplanar vectors, then show that $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}=\boldsymbol{0}$.

2. $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are non-coplanar vectors. Prove that the following four points are coplanar.

   (i) $-\boldsymbol{a}+4\boldsymbol{b}-3\boldsymbol{c}$, $3\boldsymbol{a}+2\boldsymbol{b}-5\boldsymbol{c}$, $-3\boldsymbol{a}+8\boldsymbol{b}-5\boldsymbol{c}$, $-3\boldsymbol{a}+2\boldsymbol{b}+\boldsymbol{c}$.

   (ii) $6\boldsymbol{a}+2\boldsymbol{b}-\boldsymbol{c}$, $2\boldsymbol{a}-\boldsymbol{b}+3\boldsymbol{c}$, $-\boldsymbol{a}+2\boldsymbol{b}-4\boldsymbol{c}$, $-12\boldsymbol{a}-\boldsymbol{b}-3\boldsymbol{c}$.

3. If $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are unit vectors along the positive directions of the coordinate axes, then show that the four points $4\boldsymbol{i}+5\boldsymbol{j}+\boldsymbol{k}$, $-\boldsymbol{j}-\boldsymbol{k}$, $3\boldsymbol{i}+9\boldsymbol{j}+4\boldsymbol{k}$ and $-4\boldsymbol{i}+4\boldsymbol{j}+4\boldsymbol{k}$ are coplanar.

4. If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are non-coplanar vectors, then test for the collinearity of the following points whose position vectors are given by

   (i) $\boldsymbol{a}-2\boldsymbol{b}+3\boldsymbol{c}$, $2\boldsymbol{a}+3\boldsymbol{b}-4\boldsymbol{c}$, $-7\boldsymbol{b}+10\boldsymbol{c}$

   (ii) $3\boldsymbol{a}-4\boldsymbol{b}+3\boldsymbol{c}$, $-4\boldsymbol{a}+5\boldsymbol{b}-6\boldsymbol{c}$, $4\boldsymbol{a}-7\boldsymbol{b}+6\boldsymbol{c}$

   (iii) $2\boldsymbol{a}+5\boldsymbol{b}-4\boldsymbol{c}$, $\boldsymbol{a}+4\boldsymbol{b}-3\boldsymbol{c}$, $4\boldsymbol{a}+7\boldsymbol{b}-6\boldsymbol{c}$

III. 1. In the Cartesian plane, O is the origin of the coordinate axes. A person starts at O and walks a distance of 3 units in the NORTH - EAST direction and reaches the point P. From P he walks 4 units of distance parallel to NORTH - WEST direction and reaches the point Q. Express the vector OQ in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$ (observe that $\angle XOP = 45^0$).

2. The points O, A, B, X and Y are such that **OA** $=\boldsymbol{a}$, **OB** $=\boldsymbol{b}$, **OX** $=3\boldsymbol{a}$ and **OY** $=3\boldsymbol{b}$. Find **BX** and **AY** interms of $\boldsymbol{a}$ and $\boldsymbol{b}$. Futher, if the point P divides AY in the ratio 1 : 3, then express **BP** interms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

3. In ΔOAB, E is the mid point of AB and F is a point on OA such that OF = 2FA. If C is the point of intersection of OE and BF, then find the ratios OC : CE and BC : CF.

4. The point E divides the segment PQ internally in the ratio 1 : 2 and R is any point not on the line PQ. If F is a point on QR such that QF : FR = 2 : 1, then show that EF is parallel to PR.

4.8 Vector Equations of Line and Plane

In this section we discuss the parametric vector equations of a straight line and plane which are useful in solving certain geometric problems. Hereafter P($\boldsymbol{r}$) means, P is a point with position vector $\boldsymbol{r}$.

4.8.1 Theorem: *The vector equation of the straight line passing through the point* A($\boldsymbol{a}$) *and parallel to the vector* $\boldsymbol{b}$ *is* $\boldsymbol{r} = \boldsymbol{a} + t\,\boldsymbol{b},\ t \in \mathbb{R}$.

Proof: Let P($\boldsymbol{r}$) be any point on the line (see Fig. 4.29).

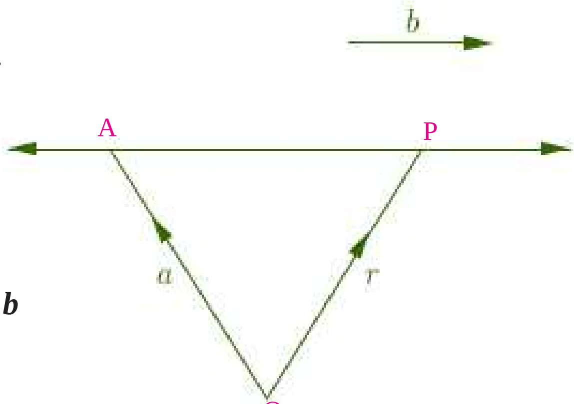


Fig. 4.29

Then **AP** and $\boldsymbol{b}$ are collinear vectors

$\therefore\ \boldsymbol{r} - \boldsymbol{a} = t\,\boldsymbol{b}$ for some $t \in \mathbb{R}$.

$\therefore\ \boldsymbol{r} = \boldsymbol{a} + t\,\boldsymbol{b}$

Conversely suppose $\boldsymbol{r} = \boldsymbol{a} + t\,\boldsymbol{b}$. Then $\boldsymbol{r} - \boldsymbol{a} = t\,\boldsymbol{b}$

$\therefore$ **AP** $= t\,\boldsymbol{b}$

$\therefore$ **AP** and $\boldsymbol{b}$ are collinear vectors.

$\therefore$ P($\boldsymbol{r}$) lies on the line.

4.8.2 Corollary: *The equation of the line passing through origin* O *and parallel to the vector* $\boldsymbol{b}$ *is* $\boldsymbol{r} = t\,\boldsymbol{b},\ t \in \mathbb{R}$.

4.8.3 Cartesian form: Cartesian equation for the line passing through A(x_1, y_1, z_1) and parallel to the vector $\boldsymbol{b} = l\,\boldsymbol{i} + m\,\boldsymbol{j} + n\,\boldsymbol{k}$ is $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$.

Fix the origin at O so that **OA** $= x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$.

If P($\boldsymbol{r}$) = (x, y, z) so that $\boldsymbol{r}$ = **OP** $= x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$ then P lies on the above line $\Leftrightarrow \boldsymbol{r} = \boldsymbol{a} + t\,\boldsymbol{b}$ for some $t \in \mathbb{R}$ (Here $\boldsymbol{a}$ means **OA**).

Now $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b} \Leftrightarrow x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = (x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}) + t(l\boldsymbol{i} + m\boldsymbol{j} + n\boldsymbol{k})$

$$\Leftrightarrow x = x_1 + t\,l,\ y = y_1 + t\,m \text{ and } z = z_1 + t\,n$$

$$\Leftrightarrow \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = t.$$

We represent these equations by $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$.

If one of l, m, n is zero, say $l = 0$, the equation becomes

$$\frac{x-x_1}{0} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = t(\neq 0).$$

This means that $x - x_1 = 0t = 0$ so that $x = x_1$. (One need not become panic on seeing 0 in the consequent as it is a ratio and not a fraction).

4.8.4 Theorem: *The vector equation of the line through the points* A ($\boldsymbol{a}$) *and* B($\boldsymbol{b}$) *is*

$$\boldsymbol{r} = (1-t)\,\boldsymbol{a} + t\,\boldsymbol{b},\ t \in \mathbb{R}.$$

Proof : Let 'O' be the origin so that **OA** = $\boldsymbol{a}$ *and* **OB** = **b**

P($\boldsymbol{r}$) is a point on the line $\Leftrightarrow$ **AP** and **AB** are collinear vectors

$\Leftrightarrow$ **AP** $= t$ **AB**, $t \in \mathbb{R}$.

$\Leftrightarrow$ $\boldsymbol{r} - \boldsymbol{a} = t(\boldsymbol{b} - \boldsymbol{a})$

$\Leftrightarrow$ $\boldsymbol{r} = (1-t)\,\boldsymbol{a} + t\,\boldsymbol{b}$.

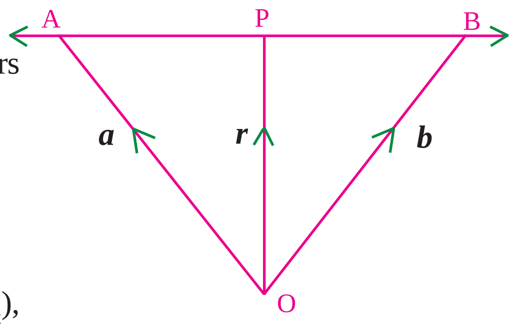


Fig. 4.30

4.8.5 Cartesian form: Let A = (x_1, y_1, z_1) and B = (x_2, y_2, z_2), P ($\boldsymbol{r}$) be a point and let

$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. Then P lies on the line AB

$$\Leftrightarrow\ x\,i + y\,j + z\,k = (1-t)\,\boldsymbol{a} + t\,\boldsymbol{b} \quad \text{for some } t \in \mathbb{R}.$$

$$\Leftrightarrow\ (x-x_1)\boldsymbol{i} + (y-y_1)\boldsymbol{j} + (z-z_1)\boldsymbol{k}$$

$$= t\left[(x_2-x_1)\boldsymbol{i} + (y_2-y_1)\boldsymbol{j} + (z_2-z_1)\boldsymbol{k}\right]$$

$$\Leftrightarrow\ x-x_1 = t(x_2-x_1),\ y-y_1 = t(y_2-y_1) \text{ and } z-z_1 = t(z_2-z_1)$$

$$\Leftrightarrow\ \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}.$$

4.8.6 Theorem: *The vector equation of the plane passing through the point* A($\boldsymbol{a}$) *and parallel to the vectors* $\boldsymbol{b}$ *and* $\boldsymbol{c}$ *is*

$$\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b} + s\boldsymbol{c};\ t, s \in \mathbb{R}.$$

Proof : Let $\boldsymbol{\sigma}$ be the plane passing through the point A($\boldsymbol{a}$) and parallel to the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ and P($\boldsymbol{r}$) be any point in $\boldsymbol{\sigma}$.

In the plane σ, through the point A, draw lines parallel to the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$. With the line segment AP as diagonal, complete the parallelogram ALPM in σ with the point L on the line parallel to $\boldsymbol{c}$ and M on the line parallel to $\boldsymbol{b}$ (see Fig. 4.31).

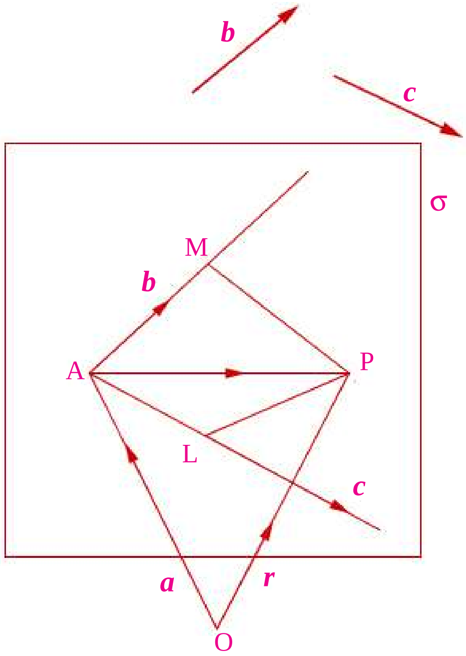


Fig. 4.31

$\therefore$ $\mathbf{AL} = s\boldsymbol{c}$, for some $s \in \mathbb{R}$ and $\mathbf{AM} = t\,\boldsymbol{b}$ for some, $t \in \mathbb{R}$.

Now $\boldsymbol{r} - \boldsymbol{a} = \mathbf{AP} = \mathbf{AL} + \mathbf{AM} = s\,\boldsymbol{c} + t\,\boldsymbol{b}$

$$\therefore \boldsymbol{r} = \boldsymbol{a} + t\,\boldsymbol{b} + s\,\boldsymbol{c}.$$

Conversely, if P is any point such that

$\boldsymbol{r} = \boldsymbol{a} + t\,\boldsymbol{b} + s\,\boldsymbol{c}$, then $\boldsymbol{r} - \boldsymbol{a} = t\,\boldsymbol{b} + s\,\boldsymbol{c}$ so that

$\mathbf{AP} = t\,\boldsymbol{b} + s\,\boldsymbol{c}$ and hence P lies in the plane σ.

4.8.7 Corollary : *The equation of the plane passing through the points* A $(\boldsymbol{a})$, B $(\boldsymbol{b})$ *and parallel to the vector* $\boldsymbol{c}$ *is* $\boldsymbol{r} = (1 - t)\,\boldsymbol{a} + t\,\boldsymbol{b} + s\,\boldsymbol{c}$, $t, s \in \mathbb{R}$.

Proof : In Theorem 4.8.6, replace the vector $\boldsymbol{b}$ with **AB**.

Then the equation of the plane is

$$\boldsymbol{r} = \boldsymbol{a} + t\,\mathbf{AB} + s\,\boldsymbol{c}$$

i.e., $\boldsymbol{r} = \boldsymbol{a} + t\,(\boldsymbol{b} - \boldsymbol{a}) + s\,\boldsymbol{c}$

i.e., $\boldsymbol{r} = (1 - t)\,\boldsymbol{a} + t\,\boldsymbol{b} + s\,\boldsymbol{c}$.

4.8.8 Corollary: *The equation of the plane passing through three noncollinear points* A $(\boldsymbol{a})$, B $(\boldsymbol{b})$ *and* C $(\boldsymbol{c})$ *is*

$$\boldsymbol{r} = (1 - t - s)\,\boldsymbol{a} + t\,\boldsymbol{b} + s\,\boldsymbol{c} \quad \text{where } t, s \in \mathbb{R}.$$

Proof : In Theorem 4.8.6, replace $\boldsymbol{b}$ with **AB** and $\boldsymbol{c}$ with **AC.**

4.8.9 Theorem: *Three points* A $(\boldsymbol{a})$, B $(\boldsymbol{b})$ *and* C $(\boldsymbol{c})$ *are collinear if and only if there exist scalars* x, y, z *(not all zero) such that* $x\,\boldsymbol{a} + y\,\boldsymbol{b} + z\,\boldsymbol{c} = \mathbf{0}$ *and* $x + y + z = 0$.

Proof : Suppose A, B and C are collinear. Then $\mathbf{AB} = \lambda\,\mathbf{BC}$ for some $\lambda \in \mathbb{R}$.

$$\Rightarrow \boldsymbol{b} - \boldsymbol{a} = \lambda\,(\boldsymbol{c} - \boldsymbol{b})$$

$$\Rightarrow \boldsymbol{a} + (-1 - \lambda)\,\boldsymbol{b} + \lambda\boldsymbol{c} = \mathbf{0}$$

Take $x = 1, y = -1 - \lambda$ and $z = \lambda$ so that $x + y + z = 0$ and $x \neq 0$.

Conversely, let x, y, z be scalars such that atleast one of them is not zero, $x\,\boldsymbol{a} + y\,\boldsymbol{b} + z\,\boldsymbol{c} = \mathbf{0}$, $x + y + z = 0$.

Suppose $z \neq 0$. Since $z = -(x + y)$ and $x\,\boldsymbol{a} + y\,\boldsymbol{b} + z\,\boldsymbol{c} = \mathbf{0}$

$$\Rightarrow x\,\boldsymbol{a} + y\,\boldsymbol{b} - (x + y)\,\boldsymbol{c} = \mathbf{0}$$

$$\therefore\ x\,(\boldsymbol{a} - \boldsymbol{c}) + y\,(\boldsymbol{b} - \boldsymbol{c}) = \mathbf{0}$$

$$\therefore\ x\,(\mathbf{CA}) + y\,(\mathbf{CB}) = \mathbf{0} \text{ and } x + y \neq 0.$$

$\therefore$ **CA** and **CB** are collinear vectors and hence the points A, B and C are collinear points.

4.8.10 Theorem: *Four points* A, B, C *and* D *with position vectors* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ *and* $\boldsymbol{d}$ *respectively are coplanar if and only if there exist scalars* x, y, z *and* u *not all zero such that*

$$x\,\boldsymbol{a} + y\,\boldsymbol{b} + z\,\boldsymbol{c} + u\,\boldsymbol{d} = \mathbf{0} \quad \textit{and} \quad x + y + z + u = 0.$$

Proof : Suppose the points A, B, C and D are coplanar.

$\therefore$ The vectors **AB**, **AC** and **AD** are coplanar.

$\therefore$ There exist scalars λ and μ such that $\mathbf{AD} = \lambda\,\mathbf{AB} + \mu\,\mathbf{AC}$.

i.e. $\boldsymbol{d} - \boldsymbol{a} = \lambda\,(\boldsymbol{b} - \boldsymbol{a}) + \mu\,(\boldsymbol{c} - \boldsymbol{a})$.

$\therefore (1 - \lambda - \mu)\,\boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c} + (-\boldsymbol{d}) = \mathbf{0}.$

Take $x = 1 - \lambda - \mu$, $y = \lambda$, $z = \mu$ and $u = -1$.

Then $x\,\boldsymbol{a} + y\,\boldsymbol{b} + z\,\boldsymbol{c} + u\,\boldsymbol{d} = \mathbf{0}$ and $x + y + z + u = 0$.

Conversely suppose that x, y, z and u are scalars such that atleast one of them is not zero, $x\,\boldsymbol{a} + y\,\boldsymbol{b} + z\,\boldsymbol{c} + u\,\boldsymbol{d} = \mathbf{0}$ and $x + y + z + u = 0$.

Suppose $u \neq 0$ so that $x + y + z = -u \neq 0$.

Now, $x\,\boldsymbol{a} + y\,\boldsymbol{b} + z\,\boldsymbol{c} + u\,\boldsymbol{d} = \mathbf{0} \Rightarrow x\,\boldsymbol{a} + y\,\boldsymbol{b} + z\,\boldsymbol{c} - (x + y + z)\,\boldsymbol{d} = 0$.

$\therefore\ x\,(\boldsymbol{a} - \boldsymbol{d}) + y\,(\boldsymbol{b} - \boldsymbol{d}) + z\,(\boldsymbol{c} - \boldsymbol{d}) = \mathbf{0}.$

$\therefore\ x\,\mathbf{DA} + y\,\mathbf{DB} + z\,\mathbf{DC} = \mathbf{0}$ and one of x, y, z is not zero. ($\because x + y + z \neq 0$)

$\therefore$ **DA**, **DB**, **DC** are coplanar vectors.

$\therefore$ The points A, B, C and D are coplanar.

4.8.11 Solved Problems

1. Problem: *In the two dimensional plane, prove by using vector methods, the equation of the line whose intercepts on the axes are 'a' and 'b' is* $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

Solution: Let A $= (a, 0)$ and B $= (0, b)$.

$\therefore$ A $= a\,\boldsymbol{i}$, B $= b\,\boldsymbol{j}$

By Theorem 4.8.4, the equation of the line AB is $\boldsymbol{r} = (1 - t)\,a\,\boldsymbol{i} + t\,(b\,\boldsymbol{j})$.

If $\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j}$, then $x = (1 - t)\,a$ and $y = t\,b$.

$$\therefore\ \frac{x}{a} + \frac{y}{b} = 1 - t + t = 1.$$

2. Problem: *Using the vector equation of the straight line passing through two points, prove that the points whose position vectors are* $\boldsymbol{a}, \boldsymbol{b}$ and $(3\boldsymbol{a} - 2\,\boldsymbol{b})$ *are collinear.*

Solution: The vector equation of the line passing through two points $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\boldsymbol{r} = (1-t)\,\boldsymbol{a} + t\boldsymbol{b}$. The line also passes through the point $3\boldsymbol{a} - 2\boldsymbol{b}$, if $3\boldsymbol{a} - 2\boldsymbol{b} = (1-t)\,\boldsymbol{a} + t\,\boldsymbol{b}$ for some scalar t. Equating the corresponding coefficients, $1 - t = 3$ and $t = -2$.

$\therefore$ The three given points are collinear.

3. Problem: *Find the equation of the line parallel to the vector* $2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ *and which passes through the point A whose position vector is* $3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$. *If P is a point on this line such that* $AP = 15$, *find the position vector of P.*

Solution: The vector equation of the given line is

$$\boldsymbol{r} = (3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) + t\,(2\,\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}), \text{ `}t\text{' being a scalar parameter.}$$

$$\text{Since } \mathbf{AP} = t\,(2\,\boldsymbol{i} - \boldsymbol{j} + 2\,\boldsymbol{k}), \text{ we have}$$

$$15 = AP = \sqrt{4t^2 + t^2 + 4t^2} = \pm 3t \Rightarrow t = \pm 5$$

$$\therefore\ \mathbf{OP} = (3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) \pm 5\,(2\,\boldsymbol{i} - \boldsymbol{j} + 2\,\boldsymbol{k})$$

$$= 13\,\boldsymbol{i} - 4\,\boldsymbol{j} + 9\,\boldsymbol{k} \text{ or } -7\,\boldsymbol{i} + 6\,\boldsymbol{j} - 11\,\boldsymbol{k}.$$

4. Problem: *Show that the line joining the pair of points* $6\boldsymbol{a} - 4\boldsymbol{b} + 4\boldsymbol{c}, -4\boldsymbol{c}$ *and the line joining the pair of points* $-\boldsymbol{a} - 2\,\boldsymbol{b} - 3\boldsymbol{c}, \boldsymbol{a} + 2\boldsymbol{b} - 5\boldsymbol{c}$ *intersect at the point* $-4\,\boldsymbol{c}$ *when* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ *are non-coplanar vectors.*

Solution: Equation of the line joining the first pair of points is

$$\boldsymbol{r} = (1-t)\,(-4\,\boldsymbol{c}) + t\,(6\boldsymbol{a} - 4\,\boldsymbol{b} + 4\,\boldsymbol{c}),\ t \in \mathbb{R}$$

$$\text{i.e., } \boldsymbol{r} = (6\,t)\,\boldsymbol{a} - (4\,t)\,\boldsymbol{b} + (8\,t - 4)\,\boldsymbol{c} \qquad \text{... (1)}$$

Equation of the line joining the second pair of points is

$$\boldsymbol{r} = (1-s)\,(-\boldsymbol{a} - 2\boldsymbol{b} - 3\boldsymbol{c}) + s\,(\boldsymbol{a} + 2\boldsymbol{b} - 5\boldsymbol{c}),\ s \in \mathbb{R}.$$

$$\text{i.e., } \boldsymbol{r} = (2s - 1)\boldsymbol{a} + (4s - 2)\boldsymbol{b} + (-2s - 3)\,\boldsymbol{c}. \qquad \text{... (2)}$$

Equating the corresponding coefficients of $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ in (1) and (2) we have $6\,t - 2s = -1$, $4\,t + 4\,s = 2$, $8\,t + 2\,s = 1$. Solving the first and second of these equations we get $t = 0$ and $s = 1/2$. These values satisfy the last equation. Substituting the value of $t = 0$ in (1) or $s = 1/2$ in (2), the point of intersection of the lines is $-4\,\boldsymbol{c}$.

5. Problem: *Find the point of intersection of the line* $\boldsymbol{r} = 2\boldsymbol{a} + \boldsymbol{b} + t(\boldsymbol{b} - \boldsymbol{c})$ *and the plane* $\boldsymbol{r} = \boldsymbol{a} + x(\boldsymbol{b} + \boldsymbol{c}) + y\,(\boldsymbol{a} + 2\,\boldsymbol{b} - \boldsymbol{c})$ *where* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ *are non-coplanar vectors.*

Solution: At the point of intersection of the line and the plane, we have

$$2\,\boldsymbol{a} + \boldsymbol{b} + t\,(\boldsymbol{b} - \boldsymbol{c}) = \boldsymbol{a} + x\,(\boldsymbol{b} + \boldsymbol{c}) + y\,(\boldsymbol{a} + 2\,\boldsymbol{b} - \boldsymbol{c}).$$

$\therefore$ On comparing the corresponding coefficients,

$$2 = 1 + y \Rightarrow y = 1$$

$$1 + t = x + 2y \Rightarrow t - x = 1$$

$$-t = x - y \Rightarrow t + x = y = 1.$$

On solving, we get $t = 1$, $x = 0$.

$\therefore$ The point of intersection $= 2\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c}$.

Exercise 4(b)

I. **1.** Find the vector equation of the line passing through the point $2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ and parallel to the vector $4\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$.

2. OABC is a parallelogram. If **OA** = $\boldsymbol{a}$ and **OC** = $\boldsymbol{c}$, find the vector equation of the side BC.

3. If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are the position vectors of the vertices A, B and C respectively of Δ ABC, then find the vector equation of the median through the vertex A.

4. Find the vector equation of the line joining the points $2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ and $-4\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$.

5. Find the vector equation of the plane passing through the points

$\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}$, $-5\boldsymbol{j} - \boldsymbol{k}$, and $-3\boldsymbol{i} + 5\boldsymbol{j}$.

6. Find the vector equation of the plane passing through the points (0,0,0), (0,5,0), and (2,0,1).

II. **1.** If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are noncoplanar find the point of intersection of the line passing through the points $2\boldsymbol{a} + 3\boldsymbol{b} - \boldsymbol{c}$, $3\boldsymbol{a} + 4\boldsymbol{b} - 2\boldsymbol{c}$ with the line joining the points $\boldsymbol{a} - 2\boldsymbol{b} + 3\boldsymbol{c}$, $\boldsymbol{a} - 6\boldsymbol{b} + 6\boldsymbol{c}$.

2. ABCD is a trapezium in which AB and CD are parallel. Prove by vector methods that the mid points of the sides AB, CD and the intersection of the diagonals are collinear.

3. In a quadrilateral ABCD, if the mid points of one pair of opposite sides and the point of intersection of the diagonals are collinear, using vector methods, prove that the quadrilateral ABCD is a trapezium.

III. **1.** Find the vector equation of the plane which passes through the points $2\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$, $2\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$ and parallel to the vector $3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$. Also find the point where this plane meets the line joining the points $2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ and $4\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$.

2. Find the vector equation of the plane passing through points $4\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$, $3\boldsymbol{i} + 7\boldsymbol{j} - 10\boldsymbol{k}$ and $2\boldsymbol{i} + 5\boldsymbol{j} - 7\boldsymbol{k}$ and show that the point $\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$ lies in the plane.

Key Concepts

- Vectors are a class of directed line segments which have both direction and magnitude.
- Vector is represented by an ordered triple (a, b, c) of real numbers. Negative of a vector **AB** is defined to be **BA.**
- The line AB is called support of the vector **AB**.
- Vectors with same support or parallel supports are called collinear vectors (parallel vectors)
- Collinear vectors are called like vectors or unlike vectors according as they have the same direction or opposite direction.
- Addition of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ using triangle law : That is, if $\mathbf{AB} = \boldsymbol{a}$ and $\mathbf{BC} = \boldsymbol{b}$, then $\boldsymbol{a} + \boldsymbol{b} = \mathbf{AC}$ and $\boldsymbol{a} - \boldsymbol{b}$ is defined as $\boldsymbol{a} + (-\boldsymbol{b})$.
- $m\boldsymbol{a}$ is the vector in the direction of $\boldsymbol{a}$ when $m > 0$ and $(-m)(-\boldsymbol{a})$ when $m < 0$, with magnitudes $m|\boldsymbol{a}|$, $(-m)|\boldsymbol{a}|$ respectively.
- $m(n\boldsymbol{a}) = (mn)\boldsymbol{a} = n(m\boldsymbol{a}) = (nm)\boldsymbol{a}$ and $m(-\boldsymbol{a}) = -m(\boldsymbol{a}) = -(m\boldsymbol{a})$.
- Position vector of a point P with reference to origin 'O' is **OP** and **AB** = **OB** – **OA**.
- Point P divides the segment AB in the ratio $m : n\,(m + n \neq 0)$ if $n\,\mathbf{AP} = m\,\mathbf{PB}$.
- If the supports are parallel to the same plane, they are called **coplanar vectors**.
- "Non coplanar vectors" means not coplanar vectors.
- Representation of a vector $\boldsymbol{r}$ in the plane determined by two non-collinear vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\boldsymbol{r} = x\boldsymbol{a} + y\boldsymbol{b}$, where x, y are unique scalars.
- Representation of any vector $\boldsymbol{r}$ in the space is $\boldsymbol{r} = x\boldsymbol{a} + y\boldsymbol{b} + z\boldsymbol{c}$ where x, y, z are unique scalars.
- If P divides the segment joining the points A($\boldsymbol{a}$) and B($\boldsymbol{b}$) in the ratio $m : n$, then the position vector of P is $\boxed{\dfrac{m\boldsymbol{b} + n\boldsymbol{a}}{m + n}}$.
- Vector equaton of the straight line passing through the point A($\boldsymbol{a}$) and parallel to the vector $\boldsymbol{b}$ is $\boxed{\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b},\ t \in \mathbb{R}}$.
- Vector equation of the straight line passing through two points A($\boldsymbol{a}$) and B($\boldsymbol{b}$) is $\boxed{\boldsymbol{r} = (1-t)\boldsymbol{a} + t\boldsymbol{b},\ t \in \mathbb{R}}$.

- Vector equation of the plane passing through a point A($\boldsymbol{a}$) and parallel to the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ is $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b} + s\boldsymbol{c},\ t, s \in \mathbb{R}$.
- Vector equation of the plane passing through three points A($\boldsymbol{a}$), B($\boldsymbol{b}$) and C($\boldsymbol{c}$) is

$$\boldsymbol{r} = (1 - t - s)\,\boldsymbol{a} + t\boldsymbol{b} + s\boldsymbol{c}.$$

- **Condition for collinearity of three points:** Three points with position vectors $\boldsymbol{a}, \boldsymbol{b}$ *and* $\boldsymbol{c}$ are collinear if and only if there exist scalars x, y and z (not all zero) such that

  $x + y + z = 0$ and $x\boldsymbol{a} + y\boldsymbol{b} + z\boldsymbol{c} = \boldsymbol{0}$.
- **Condition for coplanarity of four points:** Four points with position vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{d}$ are coplanar if and only if there exist scalars x, y, z and u (not all zero) such that $x + y + z + u = 0$ and $x\boldsymbol{a} + y\boldsymbol{b} + z\boldsymbol{c} + u\boldsymbol{d} = \boldsymbol{0}$.

Historical Note

Hermann Grassmann (1809 - 1877), the originator of calculus of extension, did a unique job in creating a new subject. In his work, which is considered as a master piece of originality, he developed the idea of an algebra in which symbols representing geometric entities such as points, lines and planes, are manipulated using certain rules.

Beginning with a collection of fundamental units $\bar{e}_1, \bar{e}_2, \bar{e}_3 \cdots$ of his algebra, he effectively defines free linear space which they generate; that is to say, he considers formal linear combinations for a hyper complex number

$$a_1\bar{e}_1 + a_2\bar{e}_2 + a_3\bar{e}_3 + \cdots$$

where $a_1, a_2, a_3 \ldots$ are real numbers and defines addition and multiplication by real numbers. He then develop the theory of linear independence in a way which is astonishingly similar to the presentation one finds in modern linear algebra texts. He goes on to prove

$$\bar{e}_1 \times \bar{e}_1 = 0$$
$$\bar{e}_1 \times \bar{e}_2 = -\bar{e}_2 \times \bar{e}_1$$
$$\bar{e}_1 \,.\, \bar{e}_1 = 1.$$

Generalisations of these operations led to newer algebras like *Clifford* algebra and Exterior algebra.

It is pertinent to say that in 1840 *Grassmann* took an examination and wrote a highly original long essay of 200 pages and introduced for the first time an analysis based on vectors, including vector addition and subtraction, vector differentiation and vector function theory.

Answers

Exercise 4(a)

I. 1. (i) $\mathbf{AL} = \mathbf{AB} + \frac{1}{2}\mathbf{AD}$, $\mathbf{AM} = \frac{1}{2}\mathbf{AB} + \mathbf{AD}$ (ii) $\lambda = \frac{3}{2}$.

2. (i) $\mathbf{DA} + \mathbf{DB} + \mathbf{DC} = \mathbf{DP} + \mathbf{DQ} + \mathbf{DR}$ (ii) $\boldsymbol{\alpha} = \mathbf{0}$

3. $\frac{1}{\sqrt{34}}(4\,\boldsymbol{i} + 3\,\boldsymbol{j} + 3\,\boldsymbol{k})$

4. $\lambda = 3$, $\mu = -6$

5. $\lambda = 3$

6. $\lambda = -1/4$

7. $\mathbf{OD} = 7\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$

8. $m = 10$, $n = 2$

9. $\frac{-1}{7}(3\boldsymbol{i} + 6\boldsymbol{j} - 2\boldsymbol{k})$

10. Equilateral triangle

11. $\cos\alpha = \frac{3}{7}, \cos\beta = -\frac{6}{7}, \cos\gamma = \frac{2}{7}$

12. $\mathrm{Cos}^{-1}\frac{2}{3}, \mathrm{Cos}^{-1}\left(-\frac{2}{3}\right), \mathrm{Cos}^{-1}\left(-\frac{1}{3}\right)$

II. 4. (i) Collinear (ii) non-collinear (iii) collinear

III. 1. $\mathbf{OQ} = \frac{1}{\sqrt{2}}(-\boldsymbol{i} + 7\,\boldsymbol{j})$

2. $\mathbf{BX} = 3\,\boldsymbol{a} - \boldsymbol{b}$, $\mathbf{AY} = 3\,\boldsymbol{b} - \boldsymbol{a}$, $\mathbf{BP} = \frac{1}{4}(3\boldsymbol{a} - \boldsymbol{b})$

3. OC : CE = 4:1 and BC : CF = 3:2

Exercise 4(b)

I. 1. $\boldsymbol{r} = (2 + 4t)\,\boldsymbol{i} + (3 - 2t)\,\boldsymbol{j} + (1 + 3t)\,\boldsymbol{k}, t \in \mathbb{R}$

2. $\boldsymbol{r} = \boldsymbol{c} + t\,\boldsymbol{a}$, $t \in \mathbb{R}$

3. $\boldsymbol{r} = (1 - t)\,\boldsymbol{a} + \dfrac{t}{2}\,(\boldsymbol{b} + \boldsymbol{c}),\ t \in \mathbb{R}$

4. $\boldsymbol{r} = 2\,(1 - 3\,t)\,\boldsymbol{i} + (1 + 2\,t\,)\,\boldsymbol{j} + (3 - 4\,t)\,\boldsymbol{k},\ t \in \mathbb{R}$

5. $\boldsymbol{r} = (1 - t - s)\,(\boldsymbol{i} - 2\,\boldsymbol{j} + 5\,\boldsymbol{k}) - t\,(5\,\boldsymbol{j} + \boldsymbol{k}) + s\,(-3\,\boldsymbol{i} + 5\,\boldsymbol{j})\,;\ t, s \in \mathbb{R}$

6. $\boldsymbol{r} = (5\,t)\,\boldsymbol{j} + s\,(2\,\boldsymbol{i} + \boldsymbol{k})\,;\ t, s \in \mathbb{R}$

II. 1. $\boldsymbol{a} + 2\boldsymbol{b}$

III. 1. $\boldsymbol{r} = (2 + 3s)\boldsymbol{i} + (4 - t - 2s)\,\boldsymbol{j} + (2 + 3t + s)\boldsymbol{k}\,;\ t, s \in \mathbb{R}, \left(\dfrac{-14}{17}, \dfrac{89}{17}, 3\right)$

2. $\boldsymbol{r} = (1 - s - t)\,(4\boldsymbol{i} - 3\,\boldsymbol{j} - \boldsymbol{k}) + s\,(3\boldsymbol{i} + 7\boldsymbol{j} - 10\boldsymbol{k}) + t\,(2\boldsymbol{i} + 5\,\boldsymbol{j} - 7\,\boldsymbol{k})\,;\ t, s \in \mathbb{R}$

Chapter 5

Product of Vectors

> *"One need not carryout operations with vectors geometrically, but can work with them algebraically"*
>
> **– *Morris Kline***

Morris Kline

(1908 - 1992)

Morris Kline *was a Professor of Mathematics, a writer on its history, philosophy and was a great teacher of mathematics, and also a popular writer of mathematical themes. His books: Mathematics : A cultural approach , and Mathematical Thought from Ancient to Modern times, are well known.*

Introduction

In Chapter 4, we studied about the addition and subtraction of vectors. We also introduced the concept of multiplication of a vector with a scalar and derived the parametric vectorial equations of straight line and plane. In this unit, we intend to introduce another algebraic operation, called the product of vectors.

Recall that product of two real numbers is a real number and product of two matrices that are compatible for multiplication, is again a matrix. But in case of functions, we may operate them in many ways, Two such operations are multiplication of functions pointwise and composition of two functions. Similarly we define two different types of products, namely, scalar (or dot) product where the resultant is a scalar and vector (or cross) product where the resultant is a vector. In the case of vectors, both the types of products have several applications in Geometry, Mechanics, Physics and Engineering.

We shall conclude this chapter by introducing the concept of scalar triple product of three vectors, explain its geometrical interpretation, indicate its use in obtaining the shortest distance between two skew lines and also discuss the vector triple product of three vectors.

5.1 Scalar or Dot product of two vectors - Geometrical Interpretation - Orthogonal Projections

5.1.1 Definition

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two vectors. The scalar (or dot) product of $\boldsymbol{a}$ and $\boldsymbol{b}$ written as $\boldsymbol{a}.\boldsymbol{b}$, is defined by

$$\boldsymbol{a}.\boldsymbol{b} = \begin{cases} 0 \text{ if one of } \boldsymbol{a},\boldsymbol{b} \text{ is } \mathbf{0} \\ |\boldsymbol{a}||\boldsymbol{b}|\cos\theta, \text{ if } \boldsymbol{a} \neq \mathbf{0} \neq \boldsymbol{b} \text{ and } \theta \text{ is the angle between } \boldsymbol{a} \text{ and } \boldsymbol{b} \end{cases}$$

5.1.2 Note

(i) $\boldsymbol{a}\,.\,\boldsymbol{b}$ *is a scalar.*

(ii) *If* $\boldsymbol{a}, \boldsymbol{b}$ *are non-zero vectors, then* $\boldsymbol{a}\,.\,\boldsymbol{b}$ *is positive or zero or negative according as the angle* θ *between* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *is acute or right or obtuse angle.*

(iii) *If* $\theta = 0$, *then* $\boldsymbol{a}\,.\,\boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|$. *In particular* $\boldsymbol{a}.\boldsymbol{a} = |\boldsymbol{a}|\,|\boldsymbol{a}|\cos 0 = |\boldsymbol{a}|^2$ *and* $\boldsymbol{a}.\boldsymbol{a}$ *is generally denoted by* $\boldsymbol{a}^2$.

(iv) *If* $\theta = \pi$, *then* $\boldsymbol{a}\,.\,\boldsymbol{b} = -\,|\boldsymbol{a}|\,|\boldsymbol{b}|$. *In particular* $\boldsymbol{a}.(-\boldsymbol{a}) = -|\boldsymbol{a}|^2$.

5.1.3 Orthogonal Projection

We introduce the concept of orthogonal projection of a vector $\boldsymbol{b}$ on a vector $\boldsymbol{a}$ and derive formulae for orthogonal projection of $\boldsymbol{b}$ on $\boldsymbol{a}$ and its magnitude, we notice that the orthogonal projection of $\boldsymbol{b}$ on $\boldsymbol{a}$ is same as the orthogonal projection of $\boldsymbol{b}$ on any vector collinear with $\boldsymbol{a}$.

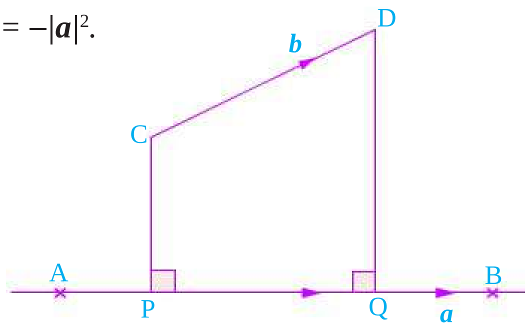


Fig. 5.1

5.1.4 Definition

*Let $\boldsymbol{a}$ = **AB** and $\boldsymbol{b}$ = **CD** be two non-zero vectors. Let P and Q be the feet of the perpendiculars drawn from C and D respectively onto the line AB (see Fig. 5.1). Then **PQ** is called **the orthogonal projection vector of $\boldsymbol{b}$ on $\boldsymbol{a}$** and the **magnitude |PQ|** is called **the magnitude of the projection of $\boldsymbol{b}$ on $\boldsymbol{a}$**. If $\boldsymbol{a} \neq \mathbf{0}$ and $\boldsymbol{b} = \mathbf{0}$, then the projection vector of $\boldsymbol{b}$ on $\boldsymbol{a}$ is defined as the zero vector.*

5.1.5 Note

(i) Some people use the word 'projection' for the projection of a vector as well as the magnitude of the projected vector. It should be understood according to the context.

(ii) The projection remains unchanged even if the supports of the vectors are replaced by parallel lines. Hence we may choose $\boldsymbol{a}$ *and* $\boldsymbol{b}$ as coinitial vectors.

5.1.6 Theorem: *The projection vector of* $\boldsymbol{b}$ *on* $\boldsymbol{a}$ *is* $\left(\dfrac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}|^2}\right)\boldsymbol{a}$ *and its magnitude is* $\dfrac{|\boldsymbol{a}.\boldsymbol{b}|}{|\boldsymbol{a}|}$.

Proof: Let $\boldsymbol{a} = \mathbf{OA}$ and $\boldsymbol{b} = \mathbf{OB}$; P be the foot of the perpendicular from B on OA and $\theta = \angle AOB$.

Case 1: θ is acute (Fig. 5.2(a)). Then

by definition, the projection of $\boldsymbol{b}$ on $\boldsymbol{a} = \mathbf{OP}$

$$= |\mathbf{OP}|\left(\frac{\boldsymbol{a}}{|\boldsymbol{a}|}\right)$$

$$= (\mathrm{OB})\cos\theta\left(\frac{\boldsymbol{a}}{|\boldsymbol{a}|}\right)$$

$$= \left(|\boldsymbol{b}|\cos\theta\right)\frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

$$= \left(|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta\right)\frac{\boldsymbol{a}}{|\boldsymbol{a}|^2}$$

$$= \frac{(\boldsymbol{a}.\boldsymbol{b})\boldsymbol{a}}{|\boldsymbol{a}|^2}.$$

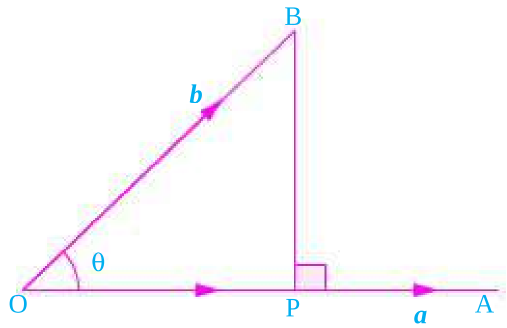


Fig. 5.2(a)

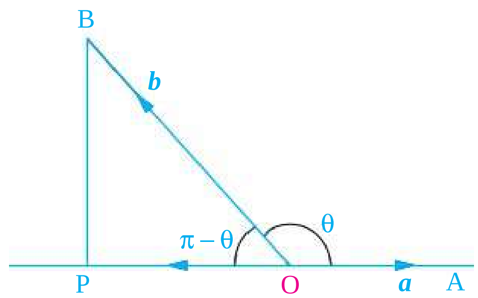


Fig. 5.2(b)

Case 2: θ is obtuse (Fig. 5.2(b)). In this case, $\mathbf{OP}$ is in the opposite direction of $\boldsymbol{a}$ and hence the angle $(\boldsymbol{b}, \mathbf{OP})$ is $\pi - \theta$.

$\therefore$ Projection of $\boldsymbol{b}$ on $\boldsymbol{a} = \mathbf{OP}$

$$= |\mathbf{OP}|\left(\frac{-\boldsymbol{a}}{|\boldsymbol{a}|}\right)$$

$$= (\mathrm{OB})\cos(\pi - \theta)\left(\frac{-\boldsymbol{a}}{|\boldsymbol{a}|}\right)$$

$$= -(\text{OB})\cos\theta\left(\frac{-\boldsymbol{a}}{|\boldsymbol{a}|}\right)$$

$$= ((\text{OB})\cos\theta)\ \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

$$= (|\boldsymbol{a}|\ |\boldsymbol{b}|\cos\theta)\ \frac{\boldsymbol{a}}{|\boldsymbol{a}|^2}$$

$$= \frac{(\boldsymbol{a}.\boldsymbol{b})}{|\boldsymbol{a}|^2}\boldsymbol{a}.$$

Case 3: When θ is a right angle, P coincides with O so that **OP** = **0** and also $\boldsymbol{a}.\boldsymbol{b} = 0$.

Hence $\textbf{OP} = \frac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}|^2}\boldsymbol{a}$.

Thus the projection vector of $\boldsymbol{b}$ on $\boldsymbol{a} = \frac{(\boldsymbol{a}.\boldsymbol{b})}{|\boldsymbol{a}|^2}\boldsymbol{a}$ and its magnitude is $\frac{|\boldsymbol{a}.\boldsymbol{b}|}{|\boldsymbol{a}|}$.

5.1.7 Definition

Let $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be non-zero vectors and* $\boldsymbol{a}$ = **OA**, $\boldsymbol{b}$ = **OB.** *Let* P *be the foot of the perpendicular from* B *on the line* OA. *Then* **OP** *is called* ***the component of*** $\boldsymbol{b}$ ***parallel to*** $\boldsymbol{a}$ *and* **PB** *is called* ***component of*** $\boldsymbol{b}$ ***perpendicular to*** $\boldsymbol{a}$ (see Fig. 5.2(a) and 5.2(b)).

Note: $\textbf{PB} = \boldsymbol{b} - \frac{(\boldsymbol{a}.\boldsymbol{b})}{|\boldsymbol{a}|^2}\boldsymbol{a}$.

If $\theta = (\boldsymbol{a}, \boldsymbol{b})$, then OP or $-$OP is called the scalar component of $\boldsymbol{b}$ on $\boldsymbol{a}$ according as $\theta \le 90^0$ or $\theta > 90^0$.

5.1.8 Geometrical interpretation of the scalar product

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two non-zero vectors and θ be the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. Let **OA** = $\boldsymbol{a}$ and **OB** = $\boldsymbol{b}$. P is the foot of the perpendicular from B on OA.

Then $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}|\ |\boldsymbol{b}|\cos\theta$

$$\therefore |\boldsymbol{a}.\boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}||\cos\theta|$$

$$= |\boldsymbol{a}||\textbf{OP}|\ \text{(See Fig.5.2 (a))}$$

= Area of the rectangle whose sides are $|\boldsymbol{a}|$ and $|\textbf{OP}|$.

5.1.9 Theorem: *Let* $\boldsymbol{a}$, $\boldsymbol{b}$ *and* $\boldsymbol{c}$ *be non-zero vectors. Then the projection of* $\boldsymbol{b}+\boldsymbol{c}$ *on* $\boldsymbol{a}$ *is equal to the sum of projections of* $\boldsymbol{b}$ *and* $\boldsymbol{c}$ *on* $\boldsymbol{a}$ *and hence*

$$\frac{\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})}{|\boldsymbol{a}|^2}\boldsymbol{a} = \frac{(\boldsymbol{a}\cdot\boldsymbol{b})}{|\boldsymbol{a}|^2}\boldsymbol{a} + \frac{(\boldsymbol{a}\cdot\boldsymbol{c})}{|\boldsymbol{a}|^2}\boldsymbol{a}.$$

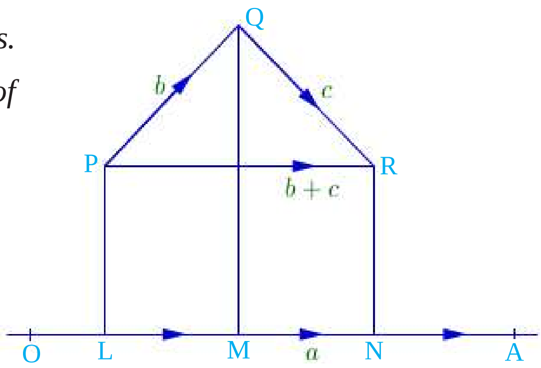


Fig. 5.3(a)

Proof : Let $\boldsymbol{a}$ = **OA**, **PQ** = $\boldsymbol{b}$, **QR** = $\boldsymbol{c}$, so that **PR** = $\boldsymbol{b}+\boldsymbol{c}$. We may assume that $\boldsymbol{b}+\boldsymbol{c} \neq 0$.

Let L, M and N be the feet of the perpendiculars drawn from P, Q and R respectively on the line OA (Fig. 5.3(a), (b)).

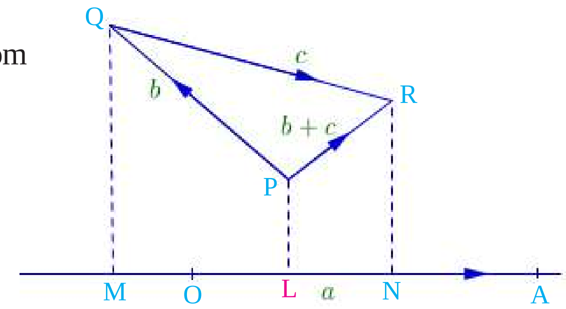


Fig. 5.3(b)

$$\frac{\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})}{|\boldsymbol{a}|^2}\boldsymbol{a} = \text{Projection of } (\boldsymbol{b}+\boldsymbol{c}) \text{ on } \boldsymbol{a}$$

$$= \mathbf{LN} = \mathbf{LM} + \mathbf{MN}$$

$$= (\text{Projection of } \boldsymbol{b} \text{ on } \boldsymbol{a}) + (\text{Projection of } \boldsymbol{c} \text{ on } \boldsymbol{a})$$

$$= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|^2}\boldsymbol{a} + \frac{\boldsymbol{a}\cdot\boldsymbol{c}}{|\boldsymbol{a}|^2}\boldsymbol{a}.$$

5.1.10 Corollary

If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are three vectors then $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\boldsymbol{c}$.

Proof : We may assume that $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{b}+\boldsymbol{c}$ are all non-zero vectors.

From 5.1.9, the projection of $(\boldsymbol{b}+\boldsymbol{c})$ on $\boldsymbol{a}$ = (projection of $\boldsymbol{b}$ on $\boldsymbol{a}$) + (projection of $\boldsymbol{c}$ on $\boldsymbol{a}$).

$$\therefore \frac{\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})}{|\boldsymbol{a}|^2}\boldsymbol{a} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|^2}\boldsymbol{a} + \frac{\boldsymbol{a}\cdot\boldsymbol{c}}{|\boldsymbol{a}|^2}\boldsymbol{a}$$

$$= \frac{(\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c})}{|\boldsymbol{a}|^2}\boldsymbol{a}$$

$$\therefore \boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\boldsymbol{c}$$

5.2 Properties of dot product

In this section we discuss some of the basic laws of dot product of two vectors.

5.2.1 Theorem: *Let* **a, b** *be two vectors. Then*

(i) $\mathbf{a} \,.\, \mathbf{b} = \mathbf{b} \,.\, \mathbf{a}$ *(commutative law)*

(ii) $(l\,\mathbf{a}) . \mathbf{b} = \mathbf{a} . (l\,\mathbf{b}) = l\,(\mathbf{a} . \mathbf{b}),\ l \in \mathbf{R}.$

(iii) $(l\,\mathbf{a}) . (m\,\mathbf{b}) = l\,m\,(\mathbf{a} . \mathbf{b}),\ l$ *and* $m \in \mathbf{R}.$

(iv) $(-\mathbf{a}) . (\mathbf{b}) = \mathbf{a} . (-\mathbf{b}) = -(\mathbf{a} . \mathbf{b})$

(v) $(-\mathbf{a}) . (-\mathbf{b}) = \mathbf{a} . \mathbf{b}.$

Proof: If one of **a, b** is a zero vector, then by the definition of dot product (i) to (v) hold.

Suppose $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$. Let $(\mathbf{a}, \mathbf{b}) = \theta$. Then

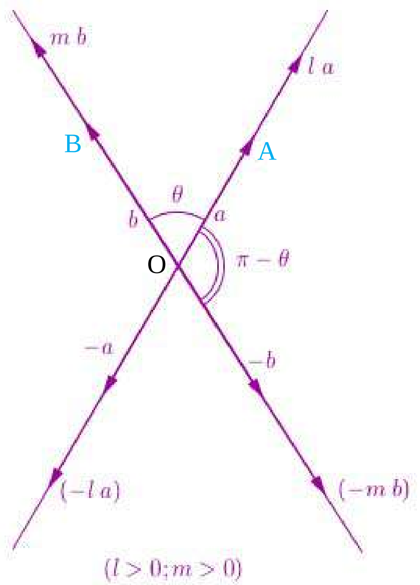


Fig. 5.4

(i) $(\mathbf{a}, \mathbf{b}) = \theta = (\mathbf{b}, \mathbf{a}).$

$\therefore\ \mathbf{b} \,.\, \mathbf{a} = |\mathbf{b}|\,|\mathbf{a}| \cos\theta = |\mathbf{a}|\,|\mathbf{b}| \cos\theta = \mathbf{a} . \mathbf{b}.$

(ii) **Case 1 :** $l > 0.$

Then $(l\,\mathbf{a}, \mathbf{b}) = (\mathbf{a}, l\,\mathbf{b}) = (\mathbf{a}, \mathbf{b}) = \theta$

$\therefore\ (l\,\mathbf{a}) . \mathbf{b} = |l\,\mathbf{a}|\ |\mathbf{b}| \cos\theta = l\,|\mathbf{a}|\ |\mathbf{b}| \cos\theta\ (\because l > 0)$

$= |\mathbf{a}|\ (l\,|\mathbf{b}| \cos\theta) = \mathbf{a} \,.\, (l\,\mathbf{b})$ and

$(l\,\mathbf{a}) . \mathbf{b} = l\ |\mathbf{a}|\ |\mathbf{b}| \cos\theta = l\,(\mathbf{a} . \mathbf{b}).$

Case 2 : $l < 0.$

$\therefore (l\,\mathbf{a}, \mathbf{b}) = (\mathbf{a}, l\,\mathbf{b}) = \pi - \theta$ (Fig. 5.4)

Now $(l\ \mathbf{a}) . \mathbf{b} = |\,l\,\mathbf{a}|\,|\,\mathbf{b}\,| \cos(\pi - \theta)$

$= (-l)\,|\,\mathbf{a}\,|\,|\,\mathbf{b}\,|\,(-\cos\theta) = l\,(\mathbf{a} . \mathbf{b})$

$\mathbf{a}\ .\,(l\ \mathbf{b}) = |\,\mathbf{a}\,|\,|l\,\mathbf{b}| \cos(\pi - \theta)$

$= |\,\mathbf{a}\,|\,(-l)\,|\,\mathbf{b}\,|\,(-\cos\theta) = l\,(\mathbf{a} . \mathbf{b})$

$\therefore\ (l\,\mathbf{a}) . \mathbf{b} = \mathbf{a} . (l\,\mathbf{b}) = l\,(\mathbf{a} . \mathbf{b})$ for all scalars 'l'.

(iii) In (ii) if we replace **b** with $m\,\mathbf{b}\ (m \in \mathbf{R})$, then

$(l\,\mathbf{a}) . (m\,\mathbf{b}) = l\,(\mathbf{a} . (m\,\mathbf{b})) = l\,(m\,\mathbf{b})) = l\,(m\,\mathbf{b} . \mathbf{a}) = l\,m\,(\mathbf{b} . \mathbf{a})$

(iv) In (ii) if we replace l with -1,

we have $(-\mathbf{a}) . \mathbf{b} = ((-1)\,\mathbf{a}) . \mathbf{b} = -1\,(\mathbf{a} . \mathbf{b}) = -(\mathbf{a} . \mathbf{b}).$

(v) In (iii) replace l with -1 and m with -1 to get the result.

5.2.2 Note: From the fact that $\mathbf{a} . \mathbf{b} = \mathbf{b} . \mathbf{a}$ and from corollary 5.1.10.

$(\mathbf{b} + \mathbf{c}) . \mathbf{a} = \mathbf{b} . \mathbf{a} + \mathbf{c} . \mathbf{a}$

and $(\mathbf{a} + \mathbf{b})^2 = \mathbf{a}^2 + \mathbf{b}^2 + 2\mathbf{a} . \mathbf{b}.$

5.3 Expression for scalar (dot) product, Angle between two Vectors

In this section, we derive formula for the dot product $\boldsymbol{a}.\boldsymbol{b}$ when $\boldsymbol{a}$ and $\boldsymbol{b}$ are expressed in terms of a right handed system ($\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$). We observe that, if $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are mutually perpendicular unit vectors, then $\boldsymbol{i}.\boldsymbol{i} = \boldsymbol{j}.\boldsymbol{j} = \boldsymbol{k}.\boldsymbol{k} = 1$ and $\boldsymbol{i}.\boldsymbol{j} = 0$, $\boldsymbol{j}.\boldsymbol{k} = 0$ and $\boldsymbol{k}.\boldsymbol{i} = 0$.

5.3.1 Theorem: *Let* ($\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$) *be the orthogonal unit triad. Let* $\boldsymbol{a} = a_1 \boldsymbol{i} + a_2 \boldsymbol{j} + a_3 \boldsymbol{k}$ *and* $\boldsymbol{b} = b_1 \boldsymbol{i} + b_2 \boldsymbol{j} + b_3 \boldsymbol{k}$ *be vectors where* a_j, b_j *are scalars for* $j = 1, 2, 3$. *Then* $\boldsymbol{a}.\boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

Proof: By Corollary 5.1.10, and Theorem 5.2.1 we have $a_1 \boldsymbol{i}.(b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k})$

$$= a_1 b_1 (\boldsymbol{i}.\boldsymbol{i}) + a_1 b_2 (\boldsymbol{i}.\boldsymbol{j}) + a_1 b_3 (\boldsymbol{i}.\boldsymbol{k})$$

$$= a_1 b_1 + 0 + 0 = a_1 b_1$$

i.e., $a_1 \boldsymbol{i}.(b_1 \boldsymbol{i} + b_2 \boldsymbol{j} + b_3 \boldsymbol{k}) = a_1 b_1$.

Similarly $a_2 \boldsymbol{j}.(b_1 \boldsymbol{i} + b_2 \boldsymbol{j} + b_3 \boldsymbol{k}) = a_2 b_2$ and $a_3 \boldsymbol{k}.(b_1 \boldsymbol{i} + b_2 \boldsymbol{j} + b_3 \boldsymbol{k}) = a_3 b_3$.

$\therefore$ Again by Corollary 5.1.10, we have $\boldsymbol{a}.\boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

5.3.2 Note

(i) *In Trigonometry, for* $|x| \le 1$, $\text{Cos}^{-1} x$ *is defined to be that angle* θ *lying between* 0 *and* π (*i.e.,* $0 \le \theta \le \pi$) *such that* $\cos\theta = x$. *Hence, if* θ *is the angle between two non-zero vectors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$, *then, from the definition of* $\boldsymbol{a}.\boldsymbol{b}$, *we have* $\theta = \text{Cos}^{-1}\left(\frac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}\right)$ *and in particular if* $\boldsymbol{a} = a_1 \boldsymbol{i} + a_2 \boldsymbol{j} + a_3 \boldsymbol{k}$ *and* $\boldsymbol{b} = b_1 \boldsymbol{i} + b_2 \boldsymbol{j} + b_3 \boldsymbol{k}$ *then*

$$\theta = \text{Cos}^{-1}\left(\frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}\right).$$

(ii) $\boldsymbol{a}, \boldsymbol{b}$ *are perpendicular to each other if and only if* $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$.

5.4 Geometrical Vector methods

In this section we study the application of dot product in proving certain geometrical results.

5.4.1 Theorem: *Angle in a semicircle is a right angle.*

Proof: Let AB be a diameter of a circle with centre O.

Let $\mathbf{OA} = \boldsymbol{a}$ so that $\mathbf{OB} = -\boldsymbol{a}$.

Let P be a point on the circle and **OP** = $\boldsymbol{r}$ (Fig. 5.5).

Then $\mathbf{PA} \cdot \mathbf{PB} = (\boldsymbol{a} - \boldsymbol{r}) \cdot (-\boldsymbol{a} - \boldsymbol{r})$

$$= -\left(\boldsymbol{a}^2 - \boldsymbol{r}^2\right)$$

$$= 0 \left(\because |\boldsymbol{a}| = |\boldsymbol{r}| = \text{radius}\right)$$

$$\therefore \angle APB = 90^0.$$

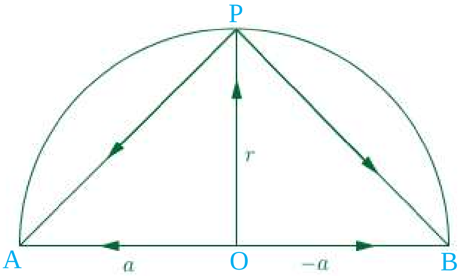


Fig. 5.5

5.4.2 Theorem: *In any triangle, the altitudes are concurrent.*

Proof: In ΔABC, let the altitudes AD and BE meet in H. Taking H as origin (Fig. 5.6),

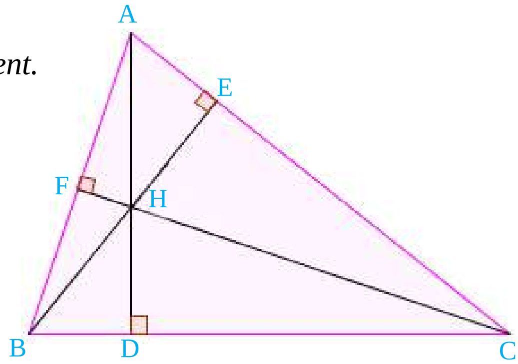


Fig. 5.6

let **HA** = $\boldsymbol{a}$, **HB** = $\boldsymbol{b}$ and **HC** = $\boldsymbol{c}$

AH is perpendicular to BC

$$\Rightarrow \mathbf{AH} \cdot \mathbf{BC} = 0$$

$$\Rightarrow -\boldsymbol{a} \cdot (\boldsymbol{c} - \boldsymbol{b}) = 0$$

$$\Rightarrow \boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{c} \cdot \boldsymbol{a} \quad \text{... (1)}$$

BH is perpendicular to AC

$$\Rightarrow \mathbf{BH} \cdot \mathbf{AC} = 0$$

$$\Rightarrow -\boldsymbol{b} \cdot (\boldsymbol{c} - \boldsymbol{a}) = 0$$

$$\Rightarrow \boldsymbol{b} \cdot \boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \quad \text{... (2)}$$

From (1) and (2), $\boldsymbol{c} \cdot \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{c}$ ... (3)

Now $\mathbf{CH} \cdot \mathbf{AB} = -\boldsymbol{c} \cdot (\boldsymbol{b} - \boldsymbol{a})$

$$= -(\boldsymbol{c} \cdot \boldsymbol{b}) + \boldsymbol{c} \cdot \boldsymbol{a} = 0 \text{ (from (3))}$$

$\therefore$ **CH** is perpendicular to **AB**.

$\therefore$ The line CF is also an altitude.

Thus the altitudes of ΔABC are concurrent.

5.4.3 Theorem: *In any triangle, the perpendicular bisectors of the sides are concurrent.*

Proof: In ΔABC, let D, E and F be the mid points of the sides BC, CA and AB respectively. Let the perpendicular lines to BC and AC at D and E respectively meet in the point 'O' (see Fig. 5.7). We show that OF is perpendicular to AB. 'O' lies on the perpendicular bisectors of the sides BC and AC.

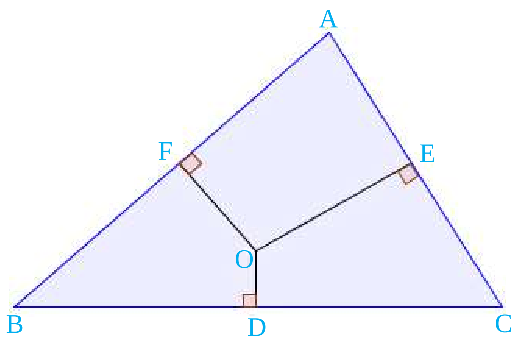


Fig. 5.7

$\therefore$ OB = OC = OA = R (say) ... (1)

Now $\mathbf{OF}.\mathbf{AB} = \frac{1}{2}(\mathbf{OB} + \mathbf{OA}).(\mathbf{OB} - \mathbf{OA})$

$= \frac{1}{2}\left(\mathbf{OB}^2 - \mathbf{OA}^2\right)$

$= \frac{1}{2}\left(R^2 - R^2\right) = 0$ (from (1))

$\therefore$ OF is perpendicular to AB.

5.4.4 Theorem (Parallelogram law)

In a parallelogram, the sum of the squares of the lengths of the diagonals is equal to sum of the squares of the lengths of its sides.

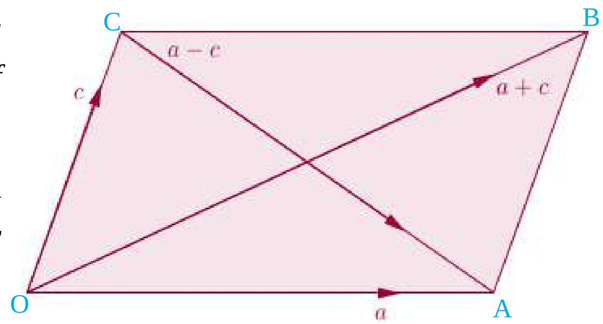


Fig. 5.8(a)

Proof: Let OABC be a parallelogram in which OB and CA are diagonals. Let $\mathbf{OA} = \boldsymbol{a}$ and $\mathbf{OC} = \boldsymbol{c}$ (see Fig. 5.8(a)).

Then $\mathbf{OB} = \boldsymbol{a} + \boldsymbol{c}$ and $\mathbf{CA} = \boldsymbol{a} - \boldsymbol{c}$

$\therefore \mathbf{OB}^2 + \mathbf{CA}^2 = |\boldsymbol{a} + \boldsymbol{c}|^2 + |\boldsymbol{a} - \boldsymbol{c}|^2$

$= \left(\boldsymbol{a}^2 + 2\boldsymbol{a}.\boldsymbol{c} + \boldsymbol{c}^2\right) + \left(\boldsymbol{a}^2 - 2\boldsymbol{a}.\boldsymbol{c} + \boldsymbol{c}^2\right)$

$= 2|\boldsymbol{a}|^2 + 2|\boldsymbol{c}|^2$

$= OA^2 + AB^2 + CB^2 + OC^2$. ($\because$ OA = BC and OC = AB)

5.4.5 Theorem: *In* ΔABC, *the length of the median through the vertex A is* $\frac{1}{2}\left(2b^2 + 2c^2 - a^2\right)^{1/2}$.

Proof: Let D be the mid point of the side BC. Take 'A' as the origin. Let $\mathbf{AB} = \boldsymbol{\alpha}$ and $\mathbf{AC} = \boldsymbol{\beta}$ so that $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \angle A$.

Since $\mathbf{AD} = \frac{\boldsymbol{\alpha} + \boldsymbol{\beta}}{2}$, we have

$4\,\mathbf{AD}^2 = \boldsymbol{\alpha}^2 + \boldsymbol{\beta}^2 + 2\boldsymbol{\alpha}.\boldsymbol{\beta}$

$= \mathbf{AB}^2 + \mathbf{AC}^2 + 2\mathbf{AB}\cdot\mathbf{AC}$

$= c^2 + b^2 + 2bc\cos A$

$= c^2 + b^2 + \left(b^2 + c^2 - a^2\right)$ [see Theorem 10.2.3]

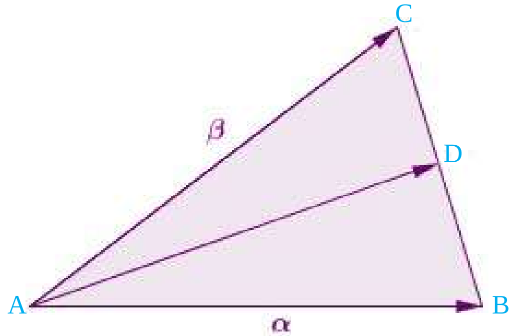


Fig. 5.8(b)

$$= 2b^2 + 2c^2 - a^2$$

$$\therefore \ \text{AD} = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}\,.$$

Note: Similarly, if BE and CF are the other medians, then $\text{BE} = \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$ and

$$\text{CF} = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}\,.$$

5.5 Vector equation of a plane - Normal form

In Chapter 4 (section 4.8), we have derived the parametric vector equations of planes. In this section, we derive vector equations of a plane, by using the scalar product of two vectors. The equation of the plane derived in this section is called the normal form.

5.5.1 Theorem: *The vector equation of the plane whose perpendicular distance from the origin is p and whose unit normal drawn from the origin towards the plane is $\mathbf{n}$, is $\mathbf{r} \cdot \mathbf{n} = p$.*

Proof : Let σ be the plane whose perpendicular distance ON from the origin 'O' is p. Let $\mathbf{n}$ be the unit vector in the direction of **ON** so that $\mathbf{ON} = p\,\mathbf{n}$. Let P be any point in the plane σ and $\mathbf{OP} = \mathbf{r}$. (see Fig. 5.9)

Since PN is perpendicular to ON, $\mathbf{ON} \cdot \mathbf{NP} = 0$

$\therefore \ (p\,\mathbf{n}) \cdot (\mathbf{r} - p\,\mathbf{n}) = 0$

$\therefore \ \mathbf{r} \cdot \mathbf{n} = p\,(\mathbf{n} \cdot \mathbf{n}) = p.$

Conversely, let P be any point and $\mathbf{r} \cdot \mathbf{n} = p$

Then $\mathbf{NP} \cdot \mathbf{n} = (\mathbf{r} - p\,\mathbf{n}) \cdot \mathbf{n}$

$= \mathbf{r} \cdot \mathbf{n} - p(\mathbf{n} \cdot \mathbf{n})$

$= \mathbf{r} \cdot \mathbf{n} - p$

$= 0.$

$\therefore$ P belongs to the plane σ.

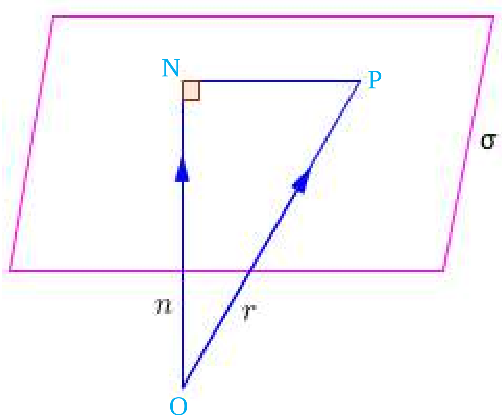


Fig. 5.9

5.5.2 Note

(i) *If the plane σ passes through the origin 'O' then $p = 0$ and hence the vector equation of σ is* $\mathbf{r} \cdot \mathbf{n} = 0$.

(ii) *If (l, m, n) are the direction cosines (see 4.1.4) of the normal to the plane σ and* $\text{P}(x, y, z)$ *is any point then,* $\text{P} \in \sigma \iff \mathbf{r} \cdot \mathbf{n} = p.$

$$\iff (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \cdot (l\,\mathbf{i} + m\,\mathbf{j} + n\,\mathbf{k}) = 0$$

$$\Leftrightarrow \quad lx + my + nz = p.$$

Thus the equation of the plane σ *is* $\boxed{lx + my + nz = p}$.

This equation of the plane is called **normal form** *in Cartesian coordinates.*

5.5.3 Theorem: *Vector equation of the plane passing through the point* A($\boldsymbol{a}$) *and perpendicular to a vector* $\boldsymbol{n}$ *is* $(\boldsymbol{r} - \boldsymbol{a}) \cdot \boldsymbol{n} = 0$.

Proof: Let P($\boldsymbol{r}$) be a point in the given plane.

Then $\overrightarrow{AP}$ is perpendicular to $\boldsymbol{n}$ and so, $(\boldsymbol{r} - \boldsymbol{a}) \cdot \boldsymbol{n} = 0$

Conversely, if P($\boldsymbol{r}$) is any point such that $(\boldsymbol{r} - \boldsymbol{a}) \cdot \boldsymbol{n} = 0$, then $\overrightarrow{AP}$ is perpendicular to $\boldsymbol{n}$.

$\therefore$ P belongs to the given plane.

5.6 Angle between two planes

We now introduce the concept of angle between two planes.

5.6.1 Definition

Let σ_1, σ_2 *be two planes,* $\boldsymbol{n}_1, \boldsymbol{n}_2$ *be the unit normals of* σ_1 *and* σ_2 *respectively. Then the angle between* σ_1 *and* σ_2 *is defined to be the angle between their normals* $\boldsymbol{n}_1$ *and* $\boldsymbol{n}_2$ (Fig 5.10(a)). *If* θ *is the angle between* σ_1 *and* σ_2 *then so is* $(180^0 - \theta)$ (Fig. 5.10(b)). *We shall take the acute angle as the angle between two planes.*

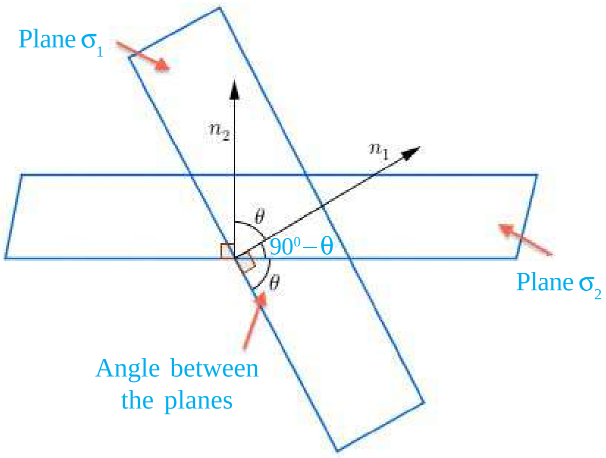


Fig. 5.10(a)

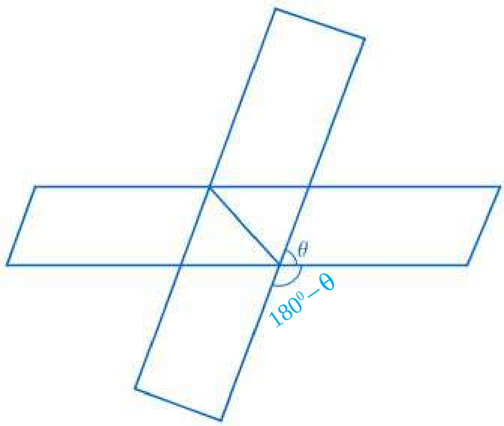


Fig. 5.10(b)

If $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are normals to the planes $\boldsymbol{r} \cdot \boldsymbol{n}_1 = d_1$ and $\boldsymbol{r} \cdot \boldsymbol{n}_2 = d_2$, and θ is the angle between the normals to the planes

$$\text{then,} \quad \cos\theta = \left| \frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|} \right| = |\boldsymbol{n}_1 \cdot \boldsymbol{n}_2|.$$

5.6.2 Note : The planes are perpendicular to each other if $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = 0$ and parallel if $\boldsymbol{n}_1$ is parallel to $\boldsymbol{n}_2$.

5.6.3 Solved Problems

1. Problem: *If* $\mathbf{a} = 6\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ *and* $\mathbf{b} = 2\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}$, *then find* $\mathbf{a}.\mathbf{b}$ *and the angle between* $\mathbf{a}$ *and* $\mathbf{b}$.

Solution: By Theorem 5.3.1, $\mathbf{a}.\mathbf{b} = 6(2) + 2(-9) + 3(6) = 12$.

Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$.

$$\therefore |\mathbf{a}| = \sqrt{6^2+2^2+3^2} = 7 \text{ and } |\mathbf{b}| = \sqrt{2^2+(-9)^2+6^2} = 11$$

$$\therefore \cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{12}{7\times 11} = \frac{12}{77}, \text{ or } \theta = \text{Cos}^{-1}\left(\frac{12}{77}\right).$$

2. Problem: *If* $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ *and* $\mathbf{b} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, *then show that* $\mathbf{a}+\mathbf{b}$ *and* $\mathbf{a}-\mathbf{b}$ *are perpendicular to each other.*

Solution: $\mathbf{a}+\mathbf{b} = 4\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{a}-\mathbf{b} = -2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$

$$\therefore (\mathbf{a}+\mathbf{b}).(\mathbf{a}-\mathbf{b}) = 4(-2) + 1(3) + (-1)(-5)$$
$$= -8 + 8 = 0.$$

$\therefore \mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ are at right angles.

3. Problem: *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be non-zero, non collinear vectors.*
If $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$, *then find the angle between* $\mathbf{a}$ *and* $\mathbf{b}$.

Solution: $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}| \Rightarrow |\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}-\mathbf{b}|^2$

$$\Rightarrow (\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b}) = (\mathbf{a}-\mathbf{b}).(\mathbf{a}-\mathbf{b})$$
$$\Rightarrow \mathbf{a}^2 + 2\mathbf{a}.\mathbf{b} + \mathbf{b}^2 = \mathbf{a}^2 - 2\mathbf{a}.\mathbf{b} + \mathbf{b}^2$$
$$\Rightarrow 4\,\mathbf{a}.\mathbf{b} = 0$$
$$\Rightarrow \mathbf{a}.\mathbf{b} = 0$$

$\therefore$ Angle between $\mathbf{a}$ and $\mathbf{b}$ is 90^0.

4. Problem: *If* $|\mathbf{a}| = 11$, $|\mathbf{b}| = 23$ *and* $|\mathbf{a}-\mathbf{b}| = 30$, *then find the angle between the vectors* $\mathbf{a}$, $\mathbf{b}$ *and also find* $|\mathbf{a}+\mathbf{b}|$.

Solution: By hypothesis $|\mathbf{a}-\mathbf{b}| = 30$.

Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$.

$$\therefore 900 = |\mathbf{a}-\mathbf{b}|^2 = \mathbf{a}^2 - 2\mathbf{a}.\mathbf{b} + \mathbf{b}^2$$
$$= 121 - (2\times 11\times 23\times \cos\theta) + 529$$
$$= 650 - (506)\cos\theta$$

$$\therefore \cos\theta = -\frac{125}{253}$$

$$\therefore \theta = \pi - \text{Cos}^{-1}\left(\frac{125}{253}\right).$$

$$|\boldsymbol{a}+\boldsymbol{b}|^2 = \boldsymbol{a}^2 + 2\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}^2 = 121 + 2 \times 11 \times 23\left(\frac{-125}{253}\right) + 529 = 400 .$$

$\therefore |\boldsymbol{a}+\boldsymbol{b}| = 20$.

5. Problem : *If* $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ *and* $\boldsymbol{b} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$, *then find the projection vector of* $\boldsymbol{b}$ *on* $\boldsymbol{a}$ *and its magnitude.*

Solution: $\boldsymbol{a}\,.\,\boldsymbol{b} = 4$, $|\boldsymbol{a}| = \sqrt{3}$. Projection vector of $\boldsymbol{b}$ *on* $\boldsymbol{a} = \dfrac{\boldsymbol{b}\,.\,\boldsymbol{a}}{|\boldsymbol{a}|^2}\,\boldsymbol{a} = \dfrac{4}{3}(\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$.

Magnitude of the projection vector $= \dfrac{|\boldsymbol{b}\,.\,\boldsymbol{a}|}{|\boldsymbol{a}|} = \dfrac{4}{\sqrt{3}}$.

6. Problem: *If* P, Q, R *and* S *are points whose position vectors are* $\boldsymbol{i} - \boldsymbol{k}$, $-\boldsymbol{i} + 2\boldsymbol{j}$, $2\boldsymbol{i} - 3\boldsymbol{k}$ *and* $3\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$ *respectively, then find the component of* **RS** *on* **PQ.**

Solution: $\mathbf{PQ} = -2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ and $\mathbf{RS} = \boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$

$|\mathbf{PQ}| = 3.$

Let $\boldsymbol{e}$ be the unit vector in the direction of **PQ**.

$$\therefore \quad \boldsymbol{e} = \frac{1}{3}(-2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}).$$

So the component of **RS** on **PQ** $= \mathbf{RS}\,.\,\boldsymbol{e} = -\dfrac{4}{3}$. (See note under 5.1.7).

7. Problem: *If the vectors* $\lambda\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k}$ *and* $2\lambda\boldsymbol{i} - \lambda\boldsymbol{j} - \boldsymbol{k}$ *are perpendicular to each other, find* λ .

Solution: By hypothesis $(\lambda\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k}).(2\lambda\boldsymbol{i} - \lambda\boldsymbol{j} - \boldsymbol{k}) = 0$

$\therefore\ 2\lambda^2 + 3\lambda - 5 = 0$

$\therefore\ (2\lambda + 5)(\lambda - 1) = 0$

$\therefore\ \lambda = \dfrac{-5}{2}$ or $1.$

8. Problem: *Let* $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = 4\boldsymbol{i} + \boldsymbol{j}$ *and* $\boldsymbol{c} = \boldsymbol{i} - 3\boldsymbol{j} - 7\boldsymbol{k}$. *Find the vector* $\boldsymbol{r}$ *such that* $\boldsymbol{r}\,.\,\boldsymbol{a} = 9$, $\boldsymbol{r}\,.\,\boldsymbol{b} = 7$ *and* $\boldsymbol{r}\,.\,\boldsymbol{c} = 6$.

Solution: Let $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

$\therefore$ By hypothesis $2x + 3y + z = 9$, $4x + y = 7$ and $x - 3y - 7z = 6$. Solving these equations for x, y and z, we have $x = 1$, $y = 3$, $z = -2$

$\therefore\ \boldsymbol{r} = \boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}.$

9. Problem: *Show that the points* $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{i} - 3\boldsymbol{j} - 5\boldsymbol{k}$ *and* $3\boldsymbol{i} - 4\boldsymbol{j} - 4\boldsymbol{k}$ *are the vertices of a right angled triangle. Also, find the other angles.*

Solution: Let the given points be A, B and C respectively (see Fig. 5.11)

Then $\mathbf{AB} = -\boldsymbol{i} - 2\boldsymbol{j} - 6\boldsymbol{k}$,

$$\mathbf{BC} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k},$$

$$\mathbf{CA} = -\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$$

$$\Rightarrow \mathbf{BC}.\mathbf{CA} = -2 - 3 + 5 = 0.$$

$$\Rightarrow \angle C = 90^0.$$

$$\cos B = \frac{\mathbf{BC}.\mathbf{BA}}{|\mathbf{BC}|\,|\mathbf{BA}|} = \sqrt{\frac{6}{41}}.$$

$$\cos A = \frac{\mathbf{AB}.\mathbf{AC}}{|\mathbf{AB}|\,|\mathbf{AC}|} = \sqrt{\frac{35}{41}}.$$

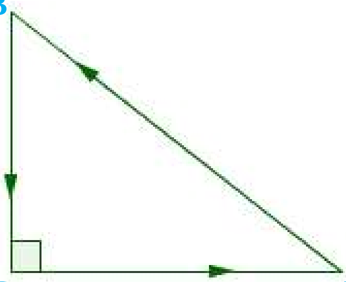


Fig. 5.11

10. Problem: *Prove that the angle* θ *between any two diagonals of a cube is given by* $\cos\theta = \frac{1}{3}$.

Solution: Without loss of generality we may assume that the cube is a unit cube.

Let $\mathbf{OA} = \boldsymbol{i}, \mathbf{OC} = \boldsymbol{j}$ and $\mathbf{OG} = \boldsymbol{k}$

(see Fig. 5.12) be coterminus edges of the cube.

$\therefore$ Diagonal $\mathbf{OE} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and diagonal

$$\mathbf{BG} = -\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}.$$

Let θ be the smaller angle between the diagonals OE and BG.

$$\text{Then } \cos\theta = \frac{|\mathbf{OE}.\mathbf{BG}|}{|\mathbf{OE}|\,|\mathbf{BG}|} = \frac{|-1-1+1|}{\sqrt{3}\sqrt{3}} = \frac{1}{3}.$$

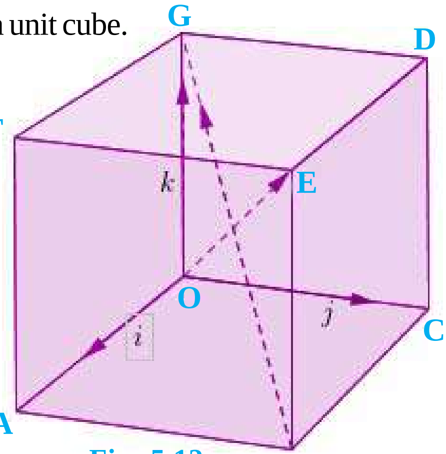


Fig. 5.12

11. Problem: *Let* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ *be non-zero mutually orthogonal vectors. If* $x\,\boldsymbol{a} + y\,\boldsymbol{b} + z\,\boldsymbol{c} = \boldsymbol{0}$, *then show that* $x = y = z = 0$.

Solution: $x\,\boldsymbol{a} + y\,\boldsymbol{b} + z\,\boldsymbol{c} = 0 \Rightarrow \boldsymbol{a}.(x\boldsymbol{a} + y\boldsymbol{b} + z\boldsymbol{c}) = 0$

$$\Rightarrow x(\boldsymbol{a}.\boldsymbol{a}) = 0$$

$$\Rightarrow x = 0 \;(\because \boldsymbol{a}.\boldsymbol{a} \neq 0).$$

Similarly $y = 0,\ z = 0.$

12. Problem: *Let* $\boldsymbol{a}, \boldsymbol{b}$ *and* $\boldsymbol{c}$ *be mutually orthogonal vectors of equal magnitudes. Prove that the vector* $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$ *is equally inclined to each of* $\boldsymbol{a}, \boldsymbol{b}$ *and* $\boldsymbol{c}$, *the angle of inclination being* $\text{Cos}^{-1}\frac{1}{\sqrt{3}}$.

Solution: Let $|\boldsymbol{a}| = |\boldsymbol{b}| = |\boldsymbol{c}| = \lambda$.

$$\text{Now, } |\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}|^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 + \boldsymbol{c}^2 + 2\sum \boldsymbol{a}.\boldsymbol{b}$$

$$= 3\lambda^2 \;(\because \boldsymbol{a}.\boldsymbol{b} = \boldsymbol{b}.\boldsymbol{c} = \boldsymbol{c}.\boldsymbol{a} = 0)$$

Let θ be the angle between $\boldsymbol{a}$ and $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$

$$\text{Then } \cos\theta = \frac{\boldsymbol{a}.(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})}{|\boldsymbol{a}||\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}|} = \frac{\boldsymbol{a}.\boldsymbol{a}}{\lambda(\lambda\sqrt{3})} = \frac{1}{\sqrt{3}}.$$

Similarly, it can be proved that $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$ inclines at an angle of $\text{Cos}^{-1}\frac{1}{\sqrt{3}}$ with $\boldsymbol{b}$ and $\boldsymbol{c}$.

13.Problem: *The vectors* $\mathbf{AB} = 3\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$ *and* $\mathbf{AD} = \boldsymbol{i} - 2\boldsymbol{k}$ *represent the adjacent sides of a parallelogram ABCD. Find the angle between the diagonals.*

Solution : From Fig. 5.13,

Diagonal $\mathbf{AC} = \mathbf{AB} + \mathbf{BC}$

$= (3\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}) + (\boldsymbol{i} - 2\boldsymbol{k})$

$= 4\boldsymbol{i} - 2\boldsymbol{j}$

Diagonal $\mathbf{BD} = -2\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$.

Let θ be the angle between **AC** and **BD**.

$$\therefore \cos\theta = \frac{\mathbf{AC}.\mathbf{BD}}{|\mathbf{AC}||\mathbf{BD}|} = \frac{-8-4}{\sqrt{20}\sqrt{24}} = -\frac{\sqrt{3}}{\sqrt{10}}.$$

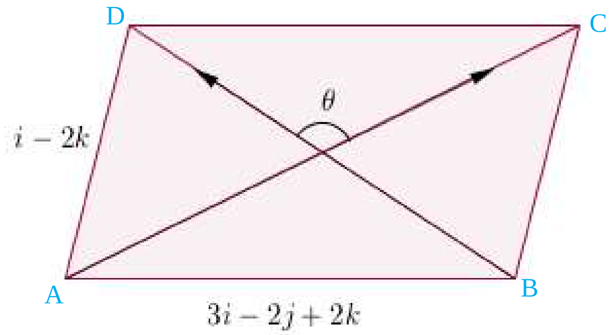


Fig. 5.13

14. Problem : *For any two vectors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$, *show that*

(i) $|\boldsymbol{a}\,.\,\boldsymbol{b}| \le |\boldsymbol{a}|\,|\boldsymbol{b}|$ (*Cauchy - Schwartz inequality*)

(ii) $|\boldsymbol{a}+\boldsymbol{b}| \le |\boldsymbol{a}| + |\boldsymbol{b}|$ (*triangle inequality*)

Solution :

(i) If $\boldsymbol{a} = \mathbf{0}$ or $\boldsymbol{b} = \mathbf{0}$, the inequalities hold trivially.

So, assume that $|\boldsymbol{a}| \ne 0 \ne |\boldsymbol{b}|$. Then $\frac{|\boldsymbol{a}\,.\boldsymbol{b}|}{|\boldsymbol{a}||\boldsymbol{b}|} = |\cos\theta| \le 1$.

Hence $|\boldsymbol{a}\,.\,\boldsymbol{b}| \le |\boldsymbol{a}|\,|\boldsymbol{b}|$.

(ii) Consider $|\boldsymbol{a}+\boldsymbol{b}|^2 = (\boldsymbol{a}+\boldsymbol{b})^2 = (\boldsymbol{a}+\boldsymbol{b})\,.\,(\boldsymbol{a}+\boldsymbol{b})$

$= \boldsymbol{a}\,.\,\boldsymbol{a} + \boldsymbol{a}\,.\,\boldsymbol{b} + \boldsymbol{b}\,.\,\boldsymbol{a} + \boldsymbol{b}\,.\,\boldsymbol{b}$

$= |\boldsymbol{a}|^2 + 2(\boldsymbol{a}\,.\,\boldsymbol{b}) + |\boldsymbol{b}|^2$, (scalar product is commutative)

$\le |\boldsymbol{a}|^2 + 2|\boldsymbol{a}\,.\,\boldsymbol{b}| + |\boldsymbol{b}|^2$ $(\because\ x \le |x|\ \ \forall x \in \mathbf{R})$

$\le |\boldsymbol{a}|^2 + 2|\boldsymbol{a}|\,|\boldsymbol{b}| + |\boldsymbol{b}|^2$ (from (i))

$= (|\boldsymbol{a}| + |\boldsymbol{b}|)^2$

Hence $|\boldsymbol{a}+\boldsymbol{b}| \le |\boldsymbol{a}| + |\boldsymbol{b}|$.

15. Problem: *Find the cartesian equation of the plane passing through the point* (–2, 1, 3) *and perpendicular to the vector* $3\boldsymbol{i} + \boldsymbol{j} + 5\boldsymbol{k}$.

Solution: Let A = $-2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ be any point P in the plane.

$\therefore \mathbf{AP} = (x+2)\boldsymbol{i} + (y-1)\boldsymbol{j} + (z-3)\boldsymbol{k}$.

$\mathbf{AP}$ is perpendicular to $3\boldsymbol{i} + \boldsymbol{j} + 5\boldsymbol{k} \Rightarrow \mathbf{AP}.(3\boldsymbol{i} + \boldsymbol{j} + 5\boldsymbol{k}) = 0$

$\Rightarrow 3(x+2) + 1(y-1) + 5(z-3) = 0$

$\Rightarrow 3x + y + 5z - 10 = 0$.

16. Problem: *Find the cartesian equation of the plane through the point* A (2, –1, –4) *and parallel to the plane* $4x - 12y - 3z - 7 = 0$.

Solution: The normal to the plane $4x - 12y - 3z - 7 = 0$ is $4\boldsymbol{i} - 12\boldsymbol{j} - 3\boldsymbol{k}$.

Hence $4\boldsymbol{i} - 12\boldsymbol{j} - 3\boldsymbol{k}$ is also normal to the required plane.

Let P = $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ be any point in the required plane.

Then $\mathbf{AP} \cdot (4\boldsymbol{i} - 12\boldsymbol{j} - 3\boldsymbol{k}) = 0$

i.e., $\left[(x-2)\boldsymbol{i} + (y+1)\boldsymbol{j} + (z+4)\boldsymbol{k}\right] . (4\boldsymbol{i} - 12\boldsymbol{j} - 3\boldsymbol{k}) = 0$

i.e., $4(x-2) - 12(y+1) - 3(z+4) = 0$

i.e., $4x - 12y - 3z = 32$.

17. Problem: *Find the angle between the planes* $2x - 3y - 6z = 5$ *and* $6x + 2y - 9z = 4$.

Solution: $\boldsymbol{n}_1 = 2\boldsymbol{i} - 3\boldsymbol{j} - 6\boldsymbol{k}$ and $\boldsymbol{n}_2 = 6\boldsymbol{i} + 2\boldsymbol{j} - 9\boldsymbol{k}$ are normals to the given planes. Let θ be the angle between the planes. Hence θ is the angle between the normals $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ (Definition 5.6.1).

$$\therefore \cos\theta = \frac{\boldsymbol{n}_1 . \boldsymbol{n}_2}{|\boldsymbol{n}_1|\,|\boldsymbol{n}_2|} = \frac{12 - 6 + 54}{\sqrt{49}\sqrt{121}} = \frac{60}{7 \times 11} = \frac{60}{77}.$$

18. Problem: *Find unit vector orthogonal to the vector* $3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}$ *and coplanar with the vectors* $2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ *and* $\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$.

Solution: Let $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{c} = 3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}$

Let $\boldsymbol{r}$ be a vector coplanar with $\boldsymbol{a}, \boldsymbol{b}$ and orthogonal to $\boldsymbol{c}$. Then

$\boldsymbol{r} = x\boldsymbol{a} + y\boldsymbol{b}$ where x, y are scalars, $\boldsymbol{r}.\boldsymbol{c} = 0$ and $|\boldsymbol{r}| = 1$.

Now, $\boldsymbol{r} = x\boldsymbol{a} + y\boldsymbol{b} = (2x+y)\boldsymbol{i} + (x-y)\boldsymbol{j} + (x+y)\boldsymbol{k}$

$\boldsymbol{r}.\boldsymbol{c} = 0 \Rightarrow 3(2x+y) + 2(x-y) + 6(x+y) = 0$

$\Rightarrow 14x + 7y = 0$

$\Rightarrow y = -2x$. ... (1)

Also $|r| = 1 \Rightarrow (2x+y)^2 + (x-y)^2 + (x+y)^2 = 1$

$\Rightarrow\ 9x^2 + x^2 = 1$ from (1)

$\Rightarrow\ x = \pm\dfrac{1}{\sqrt{10}}$

$\therefore\ \boldsymbol{r} = \pm\dfrac{1}{\sqrt{10}}(3\boldsymbol{j} - \boldsymbol{k}).$

Exercise 5(a)

I. 1. Find the angle between the vectors $\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$ and $3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$.

2. If the vectors $2\boldsymbol{i} + \lambda\boldsymbol{j} - \boldsymbol{k}$ and $4\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$ are perpendicular to each other, find λ.

3. For what values of λ, the vectors $\boldsymbol{i} - \lambda\boldsymbol{j} + 2\boldsymbol{k}$ and $8\boldsymbol{i} + 6\boldsymbol{j} - \boldsymbol{k}$ are at right angles?

4. $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - 3\boldsymbol{j} - 5\boldsymbol{k}$. Find the vector $\boldsymbol{c}$ such that $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ form the sides of a triangle.

5. Find the angle between the planes $\boldsymbol{r}.(2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) = 3$ and $\boldsymbol{r}.(3\boldsymbol{i} + 6\boldsymbol{j} + \boldsymbol{k}) = 4$.

6. Let $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ be unit vectors making angle θ. If $\dfrac{1}{2}|\boldsymbol{e}_1 - \boldsymbol{e}_2| = \sin\lambda\theta$, then find λ.

7. Let $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$. Find

(i) The projection vector of $\boldsymbol{b}$ on $\boldsymbol{a}$ and its magnitude.

(ii) The vector components of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ and perpendicular to $\boldsymbol{a}$.

8. Find the equation of the plane through the point (3, –2, 1) and perpendicular to the vector (4, 7, –4).

9. If $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$, then find the angle between $2\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a} + 2\boldsymbol{b}$.

II. 1. Find unit vector parallel to the XOY- plane and perpendicular to the vector $4\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$.

2. If $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \boldsymbol{0}$, $|\boldsymbol{a}| = 3$, $|\boldsymbol{b}| = 5$ and $|\boldsymbol{c}| = 7$, then find the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

3. If $|\boldsymbol{a}| = 2$, $|\boldsymbol{b}| = 3$ and $|\boldsymbol{c}| = 4$ and each of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ is perpendicular to the sum of the other two vectors, then find the magnitude of $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$.

4. Find the equation of the plane passing through the point $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ and perpendicular to the vector $3\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$ and the distance of this plane from the origin.

5. $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{d}$ are the position vectors of four coplanar points such that

   $(\boldsymbol{a}-\boldsymbol{d}) \cdot (\boldsymbol{b}-\boldsymbol{c}) = (\boldsymbol{b}-\boldsymbol{d}) \cdot (\boldsymbol{c}-\boldsymbol{a}) = 0$. Show that the point $\boldsymbol{d}$ represents the orthocentre of the triangle with $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ as its vertices.

III.1. Show that the points (5, –1, 1), (7, –4, 7), (1, –6, 10) and (–1, –3, 4) are the vertices of a rhombus.

2. Let $\boldsymbol{a} = 4\boldsymbol{i} + 5\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k}$ and $\boldsymbol{c} = 3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$. Find the vector which is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ whose magnitude is twenty one times the magnitude of $\boldsymbol{c}$.

3. G is the centroid of ΔABC and a, b, c are the lengths of the sides BC, CA and AB respectively.

   Prove that $a^2 + b^2 + c^2 = 3\left(OA^2 + OB^2 + OC^2\right) - 9(OG)^2$ where 'O' is any point.

4. A line makes angles $\theta_1, \theta_2, \theta_3$ and θ_4 with the diagonals of a cube.

   Show that $\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 + \cos^2\theta_4 = \dfrac{4}{3}$.

5.7 Vector product (cross product) of two vectors and properties

In this section, we recall 'Right and Left handed system' of a vector triad introduced in Chapter 4. We shall define the vector (cross) product of two vectors and study some of the properties of cross product of vectors.

5.7.1 Right handed and Left handed Systems.

Consider the three dimensional rectangular coordinate system (Fig. 5.14). In this system when the positive X-axis is rotated counter clock wise into the positive Y-axis, a right handed (standard) screw would advance in the direction of the positive Z-axis. (Fig. 5.14(i)).

In a right handed coordinate system, the thumb of the right hand points in the direction of the positive Z-axis when the fingers are curled in the direction away from the positive X-axis towards the positive Y-axis. (Fig. 5.14(ii))

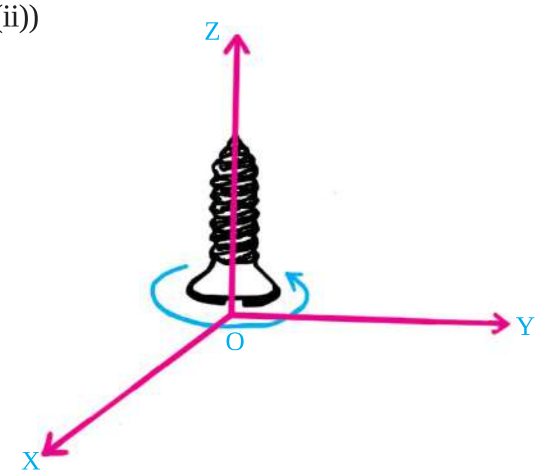


Fig. 5.14(i)

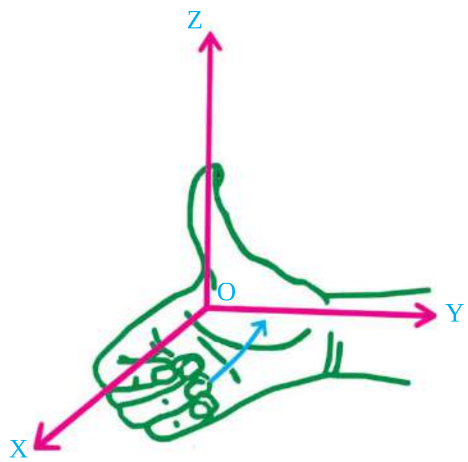


Fig. 5.14(ii)

Let O, A, B and C be points in the space such that no three of them are collinear. Let **OA** = ***a***, **OB** = ***b*** and **OC** = ***c*** (Fig. 5.15(i), (ii), (iii)). Observing from the point C, if the angle of rotation (in the counter clock wise sense) of **OA** to **OB** does not exceed 180^0, then the vector triad (***a***, ***b***, ***c***) is said to be a Right handed triad or Right handed system (Fig. 5.15(i)).

If (***a***, ***b***, ***c***) is not a right handed triad, then it is said to be a left handed triad (Fig. 5.15(ii)).

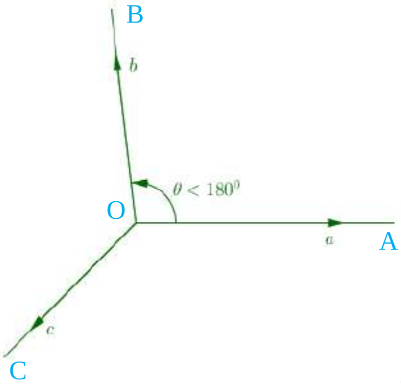


Fig. 5.15 (i)

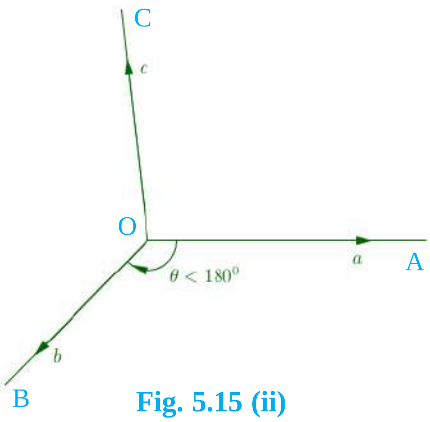


Fig. 5.15 (ii)

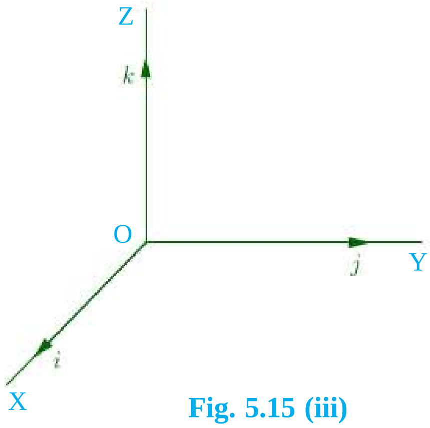


Fig. 5.15 (iii)

5.7.2 Note : (i) If (***a***, ***b***, ***c***) is a right handed (left handed) system, then the triads (***b***, ***c***, ***a***) and (***c***, ***a***, ***b***) also form right handed (left handed) systems.

(ii) If (***a***, ***b***, ***c***) is a right handed triad and ***a***, ***b***, ***c*** are mutually perpendicular to each other, then (***a***, ***b***, ***c***) is called an orthogonal triad. Thus the vector triad (***i***, ***j***, ***k***) is an orthogonal triad (Fig. 5.15(iii)).

(iii) If any two vectors in a triad are interchanged, then the system will change. For example, (***a***, ***b***, ***c***) and (***b***, ***a***, ***c***) form opposite systems.

(iv) If any vector of a system is replaced by its additive inverse, then the system changes. Thus (***a***, ***b***, ***c***) and (***a***, ***b***, –***c***) form opposite systems.

5.7.3 Definition

Let ***a*** *and* ***b*** *be non zero non collinear vectors. The* ***cross*** *(or* ***vector****)* ***product*** *of* ***a*** *and* ***b****, written as* $\mathbf{a} \times \mathbf{b}$ *(read as* ***a*** *cross* ***b****) is defined to be the vector* $\left(|\mathbf{a}|\,|\mathbf{b}|\sin\theta\right)\mathbf{n}$ *where* θ *is the angle between* ***a*** *and* ***b*** *and* ***n*** *is the unit vector perpendicular to both* ***a*** *and* ***b*** *such that (****a, b, n****) is a right handed system.*

If one of the vectors ***a, b*** *is the null vector or* ***a, b*** *are collinear vectors then the cross product* $\mathbf{a} \times \mathbf{b}$ *is defined as the null vector* ***0****.*

5.7.4 Note: *If* ***a, b*** *are non-zero and non-collinear vectors, then* $\mathbf{a} \times \mathbf{b}$ *is a vector, perpendicular to the plane determined by* ***a*** *and* ***b*** *, whose magnitude is* $|\mathbf{a}|\,|\mathbf{b}|\sin\theta$ *(observe that sin* θ *is positive).*

In the following theorem we prove that, the cross product of two non-zero non-collinear vectors does not obey the commutative law.

5.7.5 Theorem: *If* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are two vectors, then* $\boldsymbol{a} \times \boldsymbol{b} = -(\boldsymbol{b} \times \boldsymbol{a})$.

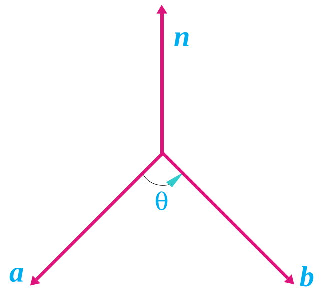


Fig. 5.16

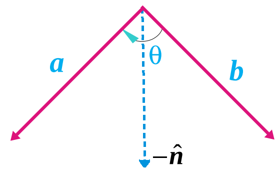


Fig. 5.17

Proof : If one of $\boldsymbol{a}, \boldsymbol{b}$ is the null vector or $\boldsymbol{a}, \boldsymbol{b}$ are collinear vectors, then $\boldsymbol{a} \times \boldsymbol{b} = \mathbf{0}$ and $\boldsymbol{b} \times \boldsymbol{a} = \mathbf{0}$ and hence $\boldsymbol{a} \times \boldsymbol{b} = -(\boldsymbol{b} \times \boldsymbol{a})$. Suppose $\boldsymbol{a}, \boldsymbol{b}$ are non–zero and non-collinear vectors. Let θ be the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ and $\boldsymbol{n}$ be the unit vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{n})$ is a right handed triad. Hence by definition $\boldsymbol{a} \times \boldsymbol{b} = (|\boldsymbol{a}||\boldsymbol{b}|\sin\theta)\boldsymbol{n}$. In this case θ is traversed from $\boldsymbol{a}$ to $\boldsymbol{b}$ (Fig. 5.16). As noted earlier (note 5.7.2 (iii) and (iv)) $(\boldsymbol{b}, \boldsymbol{a}, -\boldsymbol{n})$ is a right handed triad, i.e., θ is traversed from $\boldsymbol{b}$ to $\boldsymbol{a}$ (Fig. 5.17).

If we assume $\boldsymbol{a}$ and $\boldsymbol{b}$ to lie in the plane of the paper, then $\boldsymbol{n}$ and $-\boldsymbol{n}$ both will be perpendicular to the plane of the paper. Observe that $\boldsymbol{n}$ is directed above the paper while $-\boldsymbol{n}$ is directed below the paper.

$$\therefore \boldsymbol{b} \times \boldsymbol{a} = (|\boldsymbol{a}|\,|\boldsymbol{b}|\,\sin\theta)(-\boldsymbol{n}) = -(|\boldsymbol{a}||\boldsymbol{b}|\sin\theta)\boldsymbol{n} = -(\boldsymbol{a} \times \boldsymbol{b}).$$

Thus $(\boldsymbol{b} \times \boldsymbol{a}) = -(\boldsymbol{a} \times \boldsymbol{b})$

Note: $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{b} \times \boldsymbol{a}| = |\boldsymbol{a}|\,|\boldsymbol{b}|\sin\theta$.

5.7.6 Theorem: *Let* $\boldsymbol{a}, \boldsymbol{b}$ *be vectors and* l, m *be scalars. Then*

(i) $(-\boldsymbol{a}) \times \boldsymbol{b} = \boldsymbol{a} \times (-\boldsymbol{b}) = -(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \times \boldsymbol{a}$

(ii) $(-\boldsymbol{a}) \times (-\boldsymbol{b}) = \boldsymbol{a} \times \boldsymbol{b}$

(iii) $(l\,\boldsymbol{a}) \times \boldsymbol{b} = l\,(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a} \times (l\,\boldsymbol{b})$

(iv) $(l\,\boldsymbol{a}) \times (m\boldsymbol{b}) = l\,m\,(\boldsymbol{a} \times \boldsymbol{b})$

Proof : In the case, when one of $\boldsymbol{a}, \boldsymbol{b}$ is the null vector or they are collinear vectors or one of the scalars l, m is the zero scalar, then all the above equalities hold good. Hence we assume that $\boldsymbol{a}, \boldsymbol{b}$ are non-zero and non-collinear vectors and l, m are non-zero scalars. Let θ be the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ and $\boldsymbol{n}$ be the unit vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{n})$ is a right handed triad.

(i) Then angle between $-\boldsymbol{a}$ and $\boldsymbol{b}$ is $\pi-\theta$ (see Fig. 5.18).

From note 5.7.2(iv), the triad $(-\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{n})$ is a left handed triad and $(-\boldsymbol{a}, \boldsymbol{b}, -\boldsymbol{n})$ is a right handed triad.

$$\therefore (-\boldsymbol{a})\times\boldsymbol{b} = \left(|-\boldsymbol{a}||\boldsymbol{b}|\sin(\pi-\theta)\right)(-\boldsymbol{n})$$
$$= -\left(|\boldsymbol{a}|\,|\boldsymbol{b}|\sin\theta\right)\boldsymbol{n}$$
$$= -(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{b}\times\boldsymbol{a}.$$

Also $\quad \boldsymbol{a}\times(-\boldsymbol{b}) = -\left((-\boldsymbol{b})\times\boldsymbol{a}\right)$ (Theorem 5.7.5)
$$= -\left(-(\boldsymbol{b}\times\boldsymbol{a})\right)\left(\because (-\boldsymbol{a})\times\boldsymbol{b} = -(\boldsymbol{a}\times\boldsymbol{b})\right)$$
$$= \boldsymbol{b}\times\boldsymbol{a} = -(\boldsymbol{a}\times\boldsymbol{b})$$

Thus $(-\boldsymbol{a})\times\boldsymbol{b} = -(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{a}\times(-\boldsymbol{b}) = \boldsymbol{b}\times\boldsymbol{a}$.

(ii) $(-\boldsymbol{a})\times(-\boldsymbol{b}) = -\left[\boldsymbol{a}\times(-\boldsymbol{b})\right]$ (by (i))
$$= -\left[-(\boldsymbol{a}\times\boldsymbol{b})\right] \text{ (by (i))}$$
$$= \boldsymbol{a}\times\boldsymbol{b}.$$

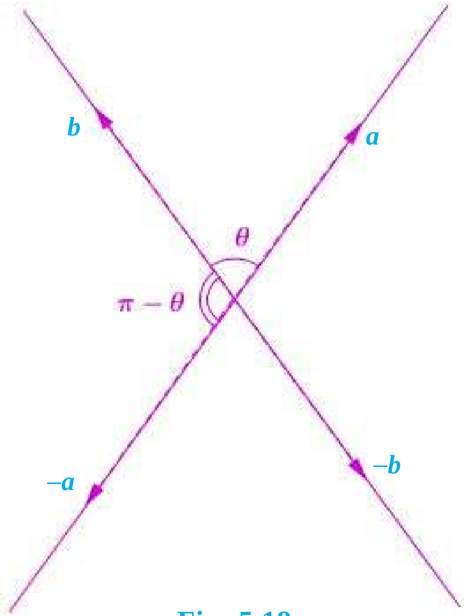


Fig. 5.18

(iii) Let $l > 0$. Then angle between $l\boldsymbol{a}$ and $\boldsymbol{b}$ is θ and $|l\boldsymbol{a}| = l|\boldsymbol{a}|$.

Further, the vector triad $(l\boldsymbol{a},\boldsymbol{b},\boldsymbol{n})$ is a right handed triad.

$$\therefore (l\boldsymbol{a})\times\boldsymbol{b} = \left(|l\boldsymbol{a}|\,|\boldsymbol{b}|\sin\theta\right)\boldsymbol{n}$$
$$= \left(l|\boldsymbol{a}||\boldsymbol{b}|\sin\theta\right)\boldsymbol{n}$$
$$= l\left[\left(|\boldsymbol{a}||\boldsymbol{b}|\sin\theta\right)\boldsymbol{n}\right]$$
$$= l\,(\boldsymbol{a}\times\boldsymbol{b}).$$

The case when $l < 0$ follows from (i) by replacing $\boldsymbol{a}$ with $l\boldsymbol{a}$ and the fact that $-l > 0$.

(iv) $(l\boldsymbol{a})\times(m\boldsymbol{b}) = lm\,(\boldsymbol{a}\times\boldsymbol{b})$ follows from (i), (ii) and (iii).

The proof of the following Theorem 5.7.7 is beyond the scope of this book and hence we assume its validity without proof.

5.7.7 Theorem (Distributive law)

If $\boldsymbol{a}, \boldsymbol{b}$ *and* $\boldsymbol{c}$ *are vectors, then* (i) $\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\times\boldsymbol{b} + \boldsymbol{a}\times\boldsymbol{c}$

(ii) $(\boldsymbol{a}+\boldsymbol{b})\times\boldsymbol{c} = \boldsymbol{a}\times\boldsymbol{c} + \boldsymbol{b}\times\boldsymbol{c}.$

Note: By assuming (i) and recalling the result that $\boldsymbol{b}\times\boldsymbol{a} = -(\boldsymbol{a}\times\boldsymbol{b})$ we get (ii).

If $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ is an orthogonal triad, then from the definition of the cross product of two vectors, it is easy to see that (i) $\boldsymbol{i}\times\boldsymbol{i} = \boldsymbol{j}\times\boldsymbol{j} = \boldsymbol{k}\times\boldsymbol{k} = \boldsymbol{0}$ and

(ii) $\boldsymbol{i}\times\boldsymbol{j} = \boldsymbol{k},\ \boldsymbol{j}\times\boldsymbol{k} = \boldsymbol{i}$ and $\boldsymbol{k}\times\boldsymbol{i} = \boldsymbol{j}$.

5.8 Vector product in (i, j, k) system

In this section, we derive formula for $\boldsymbol{a} \times \boldsymbol{b}$ when $\boldsymbol{a}$ and $\boldsymbol{b}$ are given in $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ system and deduce the formula for $|\boldsymbol{a} \times \boldsymbol{b}|$.

5.8.1 Theorem

Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ *and* $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$. *Then*

$$\boldsymbol{a} \times \boldsymbol{b} = (a_2 b_3 - a_3 b_2)\boldsymbol{i} - (a_1 b_3 - a_3 b_1)\boldsymbol{j} + (a_1 b_2 - a_2 b_1)\boldsymbol{k}$$

Proof : For proving the above formula, we use Theorem 5.7.7 and the property of cross product among $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, as mentioned at the end of Theorem 5.7.7.

$$\begin{aligned}
\boldsymbol{a} \times \boldsymbol{b} &= (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) \times (b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}) \\
&= \left[a_1 b_1 (\boldsymbol{i} \times \boldsymbol{i}) + a_1 b_2 (\boldsymbol{i} \times \boldsymbol{j}) + a_1 b_3 (\boldsymbol{i} \times \boldsymbol{k})\right] \\
&\quad + \left[\left(a_2 b_1 (\boldsymbol{j} \times \boldsymbol{i}) + a_2 b_2 (\boldsymbol{j} \times \boldsymbol{j}) + a_2 b_3 (\boldsymbol{j} \times \boldsymbol{k})\right)\right] \\
&\quad + \left[\left(a_3 b_1 (\boldsymbol{k} \times \boldsymbol{i}) + a_3 b_2 (\boldsymbol{k} \times \boldsymbol{j}) + a_3 b_3 (\boldsymbol{k} \times \boldsymbol{k})\right)\right] \\
&= \left[a_1 b_1 (\boldsymbol{0}) + a_1 b_2 \boldsymbol{k} - a_1 b_3 \boldsymbol{j}\right] + \left[-a_2 b_1 \boldsymbol{k} + a_2 b_2 (\boldsymbol{0}) + a_2 b_3 \boldsymbol{i}\right] \\
&\quad + \left[a_3 b_1 \boldsymbol{j} - a_3 b_2 \boldsymbol{i} + a_3 b_3 (\boldsymbol{0})\right]
\end{aligned}$$

$$\therefore \boldsymbol{a} \times \boldsymbol{b} = (a_2 b_3 - a_3 b_2)\boldsymbol{i} - (a_1 b_3 - a_3 b_1)\boldsymbol{j} + (a_1 b_2 - a_2 b_1)\boldsymbol{k}$$

5.8.2 Notation: Adopting the expansion of a 3×3 determinant of real matrix

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} a_1 - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} a_2 + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} a_3,$$

the above formula for $\boldsymbol{a} \times \boldsymbol{b}$ can now be expressed as

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

5.8.3 Corollary

If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$, $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ and θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, then

$$\sin\theta = \frac{\sqrt{\Sigma (a_2 b_3 - a_3 b_2)^2}}{\sqrt{\Sigma a_1^2}\sqrt{\Sigma b_1^2}}.$$

Proof: By Theorem 5.8.1, we have

$$\boldsymbol{a} \times \boldsymbol{b} = (a_2 b_3 - a_3 b_2)\boldsymbol{i} - (a_1 b_3 - a_3 b_1)\boldsymbol{j} + (a_1 b_2 - a_2 b_1)\boldsymbol{k}.$$

$\therefore \ |\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{\Sigma(a_2 b_3 - a_3 b_2)^2}$ and $|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ and $|\boldsymbol{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$.

Now, $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin\theta$, so that

$$\sin\theta = \frac{|\boldsymbol{a} \times \boldsymbol{b}|}{|\boldsymbol{a}|\,|\boldsymbol{b}|} = \frac{\sqrt{\Sigma(a_2 b_3 - a_3 b_2)^2}}{\sqrt{\Sigma a_1^2}\,\sqrt{\Sigma b_1^2}}.$$

5.8.4 Note: To determine the angle between two vectors, we use the dot product of vectors rather than the cross product, as the cross product gives value of $\sin\theta$ which is positive for $\theta \in (0, \pi)$.

5.8.5 Theorem: *For any two vectors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$,

$$|\boldsymbol{a} \times \boldsymbol{b}|^2 = (\boldsymbol{a}.\boldsymbol{a})(\boldsymbol{b}.\boldsymbol{b}) - (\boldsymbol{a}.\boldsymbol{b})^2 \quad = \boldsymbol{a}^2 \boldsymbol{b}^2 - (\boldsymbol{a}.\boldsymbol{b})^2.$$

Proof: $|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 \sin^2\theta$ where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$= |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 (1 - \cos^2\theta) = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 \cos^2\theta = (\boldsymbol{a}.\boldsymbol{a})(\boldsymbol{b}.\boldsymbol{b}) - (\boldsymbol{a}.\boldsymbol{b})^2.$$

5.8.6 Note: If $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-collinear vectors, then, unit vectors perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ *are*

$$\pm \frac{(\boldsymbol{a} \times \boldsymbol{b})}{|\boldsymbol{a} \times \boldsymbol{b}|}.$$

5.9 Vector Areas

In the following, we introduce the concept of vector area of a plane region bounded by a closed plane curve (a curve in which initial point and terminal point are the same) and find vector area of a triangle and parallelogram.

5.9.1 Definition (Vector area)

Let D *be a plane region bounded by closed curve C. Let* P_1, P_2, P_3 *be three points on* C *(taken in this order). Let* $\boldsymbol{n}$ *be the unit vector perpendicular to the region* D *such that, from the side of* $\boldsymbol{n}$, *the points* P_1, P_2 *and* P_3 *are in anticlock sense. If* A *is the area of the region* D, *then* A$\boldsymbol{n}$ *is called the* ***vector area*** *of* D. [See Fig. 5.19(a), (b)]

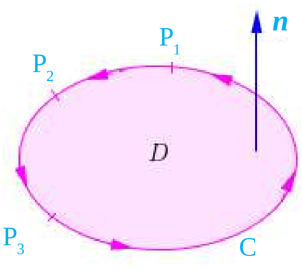


Fig. 5.19(a)

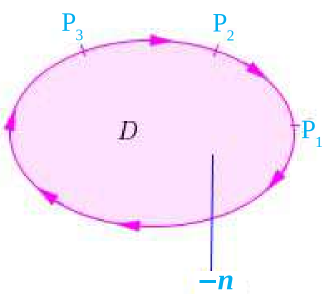


Fig. 5.19(b)

5.9.2 Note: If the points P_1, P_2 *and* P_3 are in clock sense from the side of ***n***, then the vector area is A $(-\boldsymbol{n})$. In any case the vector area of a plane region D, is either A$\boldsymbol{n}$ or A$(-\boldsymbol{n})$, so that the area is the magnitude of the vector area.

In the following theorems, we derive the vector area of a triangle and parallelogram as applications of cross product of vectors.

5.9.3 Theorem: *The vector area of* $\Delta\,\text{ABC}$ *is*

$$\frac{1}{2}(\mathbf{AB}\times\mathbf{AC})=\frac{1}{2}(\mathbf{BC}\times\mathbf{BA})=\frac{1}{2}(\mathbf{CA}\times\mathbf{CB}).$$

Proof : Let the vertices A, B and C of the triangle be described in anticlockwise sense so that the closed boundary of the plane region $\Delta\,\text{ABC}$ is $\overline{\text{BC}}\cup\overline{\text{CA}}\cup\overline{\text{AB}}$.

Let Δ be the area of $\Delta\,\text{ABC}$.

Let $\boldsymbol{n}$ be the unit vector in the direction of $\mathbf{AB}\times\mathbf{AC}$.

$$\Delta = \frac{1}{2}(\text{AB})(\text{AC})\sin\text{A}$$

$$\therefore\ \Delta\boldsymbol{n} = \frac{1}{2}(\text{AB})(\text{AC})(\sin\text{A})\boldsymbol{n}$$

$$= \frac{1}{2}|\mathbf{AB}||\mathbf{AC}|(\sin\text{A})\boldsymbol{n}$$

$$= \frac{1}{2}(\mathbf{AB}\times\mathbf{AC}).$$

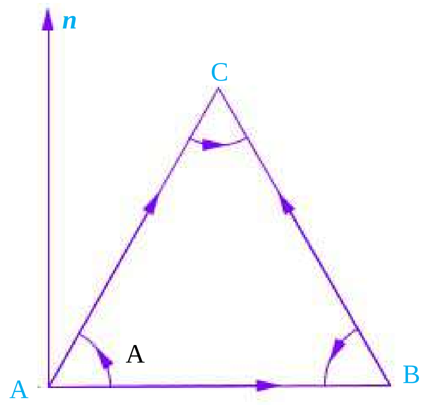


Fig. 5.20

5.9.4 Corollary: *If* ***a, b, c*** *are the position vectors of the vertices* A, B *and* C *(described in counter clockwise sense) of* $\Delta\,\text{ABC}$, *then the vector area of* $\Delta\,\text{ABC}$ *is* $\frac{1}{2}(\boldsymbol{b}\times\mathbf{c}+\mathbf{c}\times\boldsymbol{a}+\boldsymbol{a}\times\boldsymbol{b})$ *and its area is* $\frac{1}{2}|\boldsymbol{b}\times\mathbf{c}+\mathbf{c}\times\boldsymbol{a}+\boldsymbol{a}\times\boldsymbol{b}|$.

Proof : From Theorem 5.9.3, the vector area of

$$\Delta\,\text{ABC} = \frac{1}{2}(\mathbf{AB}\times\mathbf{AC})$$

$$= \frac{1}{2}[(\boldsymbol{b}-\boldsymbol{a})\times(\boldsymbol{c}-\boldsymbol{a})]$$

$$= \frac{1}{2}[\boldsymbol{b}\times\boldsymbol{c}+\boldsymbol{b}\times(-\boldsymbol{a})+(-\boldsymbol{a})\times\boldsymbol{c}+(-\boldsymbol{a})\times(-\boldsymbol{a})]$$

$$= \frac{1}{2}[\boldsymbol{b}\times\boldsymbol{c}-(\boldsymbol{b}\times\boldsymbol{a})-(\boldsymbol{a}\times\boldsymbol{c})+\mathbf{0}]$$

$$= \frac{1}{2}[\boldsymbol{b}\times\boldsymbol{c}+\boldsymbol{c}\times\boldsymbol{a}+\boldsymbol{a}\times\boldsymbol{b}]$$

Area of $\Delta\,\text{ABC} = \Delta = |\Delta\boldsymbol{n}| = \frac{1}{2}|\boldsymbol{b}\times\boldsymbol{c}+\boldsymbol{c}\times\boldsymbol{a}+\boldsymbol{a}\times\boldsymbol{b}|$.

5.9.5 Note: Since vector area of a plane region D is a vector quantity perpendicular to the plane of D, it follows that, the vector $(\boldsymbol{b}\times\boldsymbol{c})+(\boldsymbol{c}\times\boldsymbol{a})+(\boldsymbol{a}\times\boldsymbol{b})$ is perpendicular to the plane of the ΔABC where ***a, b, c*** are the position vectors of A, B and C respectively.

5.9.6 Theorem (Vector area of a parallelogram): *Let ABCD be a parallelogram with vertices A, B, C and D described in counter clockwise sense. Then, the vector area of ABCD in terms of the diagonals AC and BD is* $\frac{1}{2}(\mathbf{AC}\times\mathbf{BD})$.

Proof : $\frac{1}{2}(\mathbf{AC}\times\mathbf{BD})$

$$= \frac{1}{2}(\mathbf{AB}+\mathbf{BC})\times(\mathbf{BA}+\mathbf{AD})$$

$$= \frac{1}{2}\left[\mathbf{AB}\times\mathbf{BA}+\mathbf{AB}\times\mathbf{AD}+\mathbf{BC}\times\mathbf{BA}+\mathbf{BC}\times\mathbf{AD}\right]$$

$$= \frac{1}{2}\left[\mathbf{AB}\times\mathbf{AD}+(-\mathbf{CB})\times\mathbf{BA}\right]$$

$$= \frac{1}{2}\left[\mathbf{AB}\times\mathbf{AD}+(-\mathbf{CB})\times\mathbf{CD}\right]\ (\because\ \mathbf{BA}=\mathbf{CD})$$

$$= \frac{1}{2}(\mathbf{AB}\times\mathbf{AD})+\frac{1}{2}(\mathbf{CD}\times\mathbf{CB})$$

= Vector area of ΔABD + vector area of ΔCDB

= Vector area of ABCD.

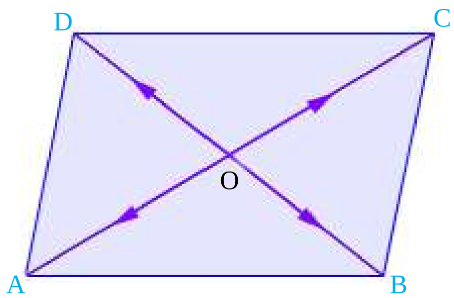


Fig. 5.21

5.9.7 Note

(i) In fact, the vector area of any plane quadrilateral ABCD in terms of the diagonals AC and BD is $\frac{1}{2}(\mathbf{AC}\times\mathbf{BD})$.

(ii) The area of the quadrilateral ABCD is $\frac{1}{2}|\mathbf{AC}\times\mathbf{BD}|$.

(iii) The vector area of a parallelogram with ***a*** and ***b*** as adjacent sides is $\boldsymbol{a}\times\boldsymbol{b}$ and the area is $|\boldsymbol{a}\times\boldsymbol{b}|$.

5.9.8 Theorem: *Let* $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$ *be a non-coplanar vector triad,* $\boldsymbol{\alpha} = l_1\boldsymbol{a}+l_2\boldsymbol{b}+l_3\boldsymbol{c}$ *and* $\boldsymbol{\beta} = m_1\boldsymbol{a}+m_2\boldsymbol{b}+m_3\boldsymbol{c}$. *Then*

$$\boldsymbol{\alpha}\times\boldsymbol{\beta} = \begin{vmatrix} \boldsymbol{b}\times\boldsymbol{c} & \boldsymbol{c}\times\boldsymbol{a} & \boldsymbol{a}\times\boldsymbol{b} \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix}.$$

Proof: Using the distributive law of cross product over vector addition (Theorem 5.7.7) we have

$$\boldsymbol{\alpha}\times\boldsymbol{\beta} = l_1 m_2(\boldsymbol{a}\times\boldsymbol{b})+l_1 m_3(\boldsymbol{a}\times\boldsymbol{c})+l_2 m_1(\boldsymbol{b}\times\boldsymbol{a})+l_2 m_3(\boldsymbol{b}\times\boldsymbol{c})+l_3 m_1(\boldsymbol{c}\times\boldsymbol{a})+l_3 m_2(\boldsymbol{c}\times\boldsymbol{b})$$

$$= (l_2 m_3 - l_3 m_2)(\boldsymbol{b} \times \boldsymbol{c}) - (l_1 m_3 - l_3 m_1)(\boldsymbol{c} \times \boldsymbol{a}) + (l_1 m_2 - l_2 m_1)(\boldsymbol{a} \times \boldsymbol{b})$$

$$(\because \ \boldsymbol{b} \times \boldsymbol{a} = -(\boldsymbol{a} \times \boldsymbol{b}),\ \boldsymbol{c} \times \boldsymbol{b} = -(\boldsymbol{b} \times \boldsymbol{c}) \text{ and } \boldsymbol{a} \times \boldsymbol{c} = -(\boldsymbol{c} \times \boldsymbol{a}))$$

$$= \begin{vmatrix} \boldsymbol{b}\times\boldsymbol{c} & \boldsymbol{c}\times\boldsymbol{a} & \boldsymbol{a}\times\boldsymbol{b} \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix}.$$

Note : In the above theorem, if we take $\boldsymbol{a} = \boldsymbol{i}, \boldsymbol{b} = \boldsymbol{j}$ and $\boldsymbol{c} = \boldsymbol{k}$ such that $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ is a right handed system, then we obtain the formula for $\alpha \times \beta$ as in 5.8.2.

5.9.9 Solved Problems

1. Problem: *If $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k}$, $\boldsymbol{b} = -\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$ then find $\boldsymbol{a} \times \boldsymbol{b}$ and unit vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.*

Solution: $\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -3 & 5 \\ -1 & 4 & 2 \end{vmatrix} = -26\boldsymbol{i} - 9\boldsymbol{j} + 5\boldsymbol{k}$

The unit vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$

$$= \pm \frac{\boldsymbol{a} \times \boldsymbol{b}}{|\boldsymbol{a} \times \boldsymbol{b}|} = \pm \frac{1}{\sqrt{782}}(-26\boldsymbol{i} - 9\boldsymbol{j} + 5\boldsymbol{k}).$$

2. Problem: *If $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k}$, $\boldsymbol{b} = -\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$, then find $(\boldsymbol{a} + \boldsymbol{b}) \times (\boldsymbol{a} - \boldsymbol{b})$ and unit vector perpendicular to both $\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a} - \boldsymbol{b}$.*

Solution: $(\boldsymbol{a} + \boldsymbol{b}) \times (\boldsymbol{a} - \boldsymbol{b}) = -(\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{b} \times \boldsymbol{a})$

$$= -2(\boldsymbol{a} \times \boldsymbol{b}) = -2(-26\boldsymbol{i} - 9\boldsymbol{j} + 5\boldsymbol{k}) \text{ (see problem 1)}$$

$$= 52\boldsymbol{i} + 18\boldsymbol{j} - 10\boldsymbol{k}$$

Unit vector perpendicular to both $\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a} - \boldsymbol{b}$

$$= \pm \frac{1}{\sqrt{782}}(26\boldsymbol{i} + 9\boldsymbol{j} - 5\boldsymbol{k}).$$

Remark: In problems 1 and 2, you find that the unit vectors perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ are same as the unit vectors perpendicular to both $\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a} - \boldsymbol{b}$. Give justification.

3. Problem: *Find the area of the parallelogram for which the vectors $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j}$ and $\boldsymbol{b} = 3\boldsymbol{i} - \boldsymbol{k}$ are adjacent sides.*

Solution: The vector area of the given parallelogram is

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -3 & 0 \\ 3 & 0 & -1 \end{vmatrix} = 3\boldsymbol{i} + 2\boldsymbol{j} + 9\boldsymbol{k}.$$

$$\therefore \text{ Area } = |\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{94}.$$

4. Problem: *If* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ *and* $\boldsymbol{d}$ *are vectors such that* $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c} \times \boldsymbol{d}$ *and* $\boldsymbol{a} \times \boldsymbol{c} = \boldsymbol{b} \times \boldsymbol{d}$*, then show that the vectors* $\boldsymbol{a} - \boldsymbol{d}$ *and* $\boldsymbol{b} - \boldsymbol{c}$ *are parallel.*

Solution: $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c} \times \boldsymbol{d}$, $\boldsymbol{a} \times \boldsymbol{c} = \boldsymbol{b} \times \boldsymbol{d}$. On subtraction, $\boldsymbol{a} \times (\boldsymbol{b} - \boldsymbol{c}) = (\boldsymbol{c} - \boldsymbol{b}) \times \boldsymbol{d}$

$$= \boldsymbol{d} \times (\boldsymbol{b} - \boldsymbol{c})$$

$$\therefore (\boldsymbol{a} - \boldsymbol{d}) \times (\boldsymbol{b} - \boldsymbol{c}) = \boldsymbol{0}$$

$\therefore \boldsymbol{a} - \boldsymbol{d}$ and $\boldsymbol{b} - \boldsymbol{c}$ are parallel vectors.

5. Problem: *If* $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$ *and* $\boldsymbol{b} = 3\boldsymbol{i} + 5\boldsymbol{j} - \boldsymbol{k}$ *are two sides of a triangle, then find its area.*

Solution: Area of the triangle is equal to half of the area of the parallelogram for which $\boldsymbol{a}$ and $\boldsymbol{b}$ are adjacent sides

$$= \frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}|$$

$$\text{But } \boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 3 \\ 3 & 5 & -1 \end{vmatrix} = -17\boldsymbol{i} + 10\boldsymbol{j} - \boldsymbol{k}.$$

$$\therefore \text{Area of the triangle} = \frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}| = \frac{\sqrt{390}}{2}.$$

6. Problem: *In* ΔABC*, if* $\mathbf{BC} = \boldsymbol{a}$, $\mathbf{CA} = \boldsymbol{b}$ *and* $\mathbf{AB} = \boldsymbol{c}$ *then show that* $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{c} \times \boldsymbol{a}$.

Solution: $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \mathbf{BC} + \mathbf{CA} + \mathbf{AB} = \mathbf{BB} = \mathbf{0}$

$$\therefore \boldsymbol{a} + \boldsymbol{b} = -\boldsymbol{c}$$

$$\therefore \boldsymbol{a} \times (\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{a} \times (-\boldsymbol{c})$$

$$\therefore \boldsymbol{a} \times \boldsymbol{b} = -(\boldsymbol{a} \times \boldsymbol{c}) = \boldsymbol{c} \times \boldsymbol{a}.$$

Also $(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{b} = (-\boldsymbol{c}) \times \boldsymbol{b}$

$$\therefore \boldsymbol{a} \times \boldsymbol{b} = -(\boldsymbol{c} \times \boldsymbol{b}) = \boldsymbol{b} \times \boldsymbol{c}$$

$$\therefore \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c} \times \boldsymbol{a}.$$

7. Problem: *Let* $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ *and* $\boldsymbol{b} = 3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$. *If* θ *is the angle between* $\boldsymbol{a}$ *and* $\boldsymbol{b}$*, then find* $\sin\theta$.

Solution: $\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix} = -3\boldsymbol{i} + 5\boldsymbol{j} + 11\boldsymbol{k}$ and $|\boldsymbol{a}| = \sqrt{6}$, $|\boldsymbol{b}| = \sqrt{26}$ and

$|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{155}$.

$$\text{Now } \sin\theta = \frac{|\boldsymbol{a} \times \boldsymbol{b}|}{|\boldsymbol{a}|\,|\boldsymbol{b}|} = \frac{\sqrt{155}}{\sqrt{6}\sqrt{26}} = \sqrt{\frac{155}{156}}.$$

8. Problem: *Let* $\boldsymbol{a}, \boldsymbol{b}$ *and* $\boldsymbol{c}$ *be such that* $\boldsymbol{c} \neq \mathbf{0}$, $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c}$, $\boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{a}$. *Show that* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ *are pair wise orthogonal vectors and* $|\boldsymbol{b}| = 1, |\boldsymbol{c}| = |\boldsymbol{a}|$.

Solution: $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c} \Rightarrow \boldsymbol{c}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

$\boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{a} \Rightarrow \boldsymbol{a}$ is perpendicular to both $\boldsymbol{b}$ and $\boldsymbol{c}$.

$\therefore \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are mutually orthogonal vectors

$$\therefore |\boldsymbol{c}| = |\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin 90^0 = |\boldsymbol{a}||\boldsymbol{b}| \quad \text{... (1)}$$

$$|\boldsymbol{a}| = |\boldsymbol{b} \times \boldsymbol{c}| = |\boldsymbol{b}||\boldsymbol{c}| \sin 90^0 = |\boldsymbol{b}||\boldsymbol{c}| \quad \text{... (2)}$$

From (1) and (2), $|\boldsymbol{c}||\boldsymbol{a}| = |\boldsymbol{c}||\boldsymbol{a}||\boldsymbol{b}|^2$

$\therefore |\boldsymbol{b}| = 1$ and from (1), $|\boldsymbol{c}| = |\boldsymbol{a}|$.

9. Problem: *Let* $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j}$. *If* $\boldsymbol{c}$ *is a vector such that* $\boldsymbol{a} \cdot \boldsymbol{c} = |\boldsymbol{c}|$, $|\boldsymbol{c} - \boldsymbol{a}| = 2\sqrt{2}$ *and the angle between* $\boldsymbol{a} \times \boldsymbol{b}$ *and* $\boldsymbol{c}$ *is* 30^0, *then find the value of* $|(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}|$.

Solution: $|\boldsymbol{a}| = 3$, $|\boldsymbol{b}| = \sqrt{2}$ and $\boldsymbol{a}.\boldsymbol{c} = |\boldsymbol{c}|$.

$$2\sqrt{2} = |\boldsymbol{c} - \boldsymbol{a}| \Rightarrow 8 = |\boldsymbol{c} - \boldsymbol{a}|^2 = |\boldsymbol{c}|^2 + |\boldsymbol{a}|^2 - 2(\boldsymbol{a}.\boldsymbol{c})$$

$$\therefore 8 = |\boldsymbol{c}|^2 + 9 - 2|\boldsymbol{c}|$$

$$\therefore (|\boldsymbol{c}| - 1)^2 = 0$$

$$\therefore |\boldsymbol{c}| = 1.$$

$$\text{Now } \boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}.$$

$$\therefore |(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}| = |\boldsymbol{a} \times \boldsymbol{b}|\,|\boldsymbol{c}| \sin 30^0 = 3(1)\frac{1}{2} = \frac{3}{2}.$$

10. Problem: *Let* $\boldsymbol{a}, \boldsymbol{b}$ *be two non-collinear unit vectors. If* $\boldsymbol{\alpha} = \boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{b}$ *and* $\boldsymbol{\beta} = \boldsymbol{a} \times \boldsymbol{b}$, *then show that* $|\boldsymbol{\beta}| = |\boldsymbol{\alpha}|$.

Solution: $|\beta|^2 = |\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (\boldsymbol{a}.\boldsymbol{b})^2$ (see Theorem 5.8.5)

$= 1 - \cos^2\theta = \sin^2\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$|\alpha|^2 = |\boldsymbol{a}|^2 + (\boldsymbol{a}.\boldsymbol{b})^2 |\boldsymbol{b}|^2 - 2(\boldsymbol{a}.\boldsymbol{b})^2$$

$$= 1 + \cos^2\theta - 2\cos^2\theta = \sin^2\theta$$

$$\therefore |\beta| = |\alpha|.$$

11. Problem: *A non-zero vector* $\boldsymbol{a}$ *is parallel to the line of intersection of the plane determined by the vectors* $\boldsymbol{i}$, $\boldsymbol{i}+\boldsymbol{j}$ *and the plane determined by the vectors* $\boldsymbol{i}-\boldsymbol{j}$, $\boldsymbol{i}+\boldsymbol{k}$. *Find the angle between* $\boldsymbol{a}$ *and the vector* $\boldsymbol{i}-2\boldsymbol{j}+2\boldsymbol{k}$.

Solution: Let l be the line of intersection of the planes determined by the pairs $\boldsymbol{i}$, $\boldsymbol{i}+\boldsymbol{j}$ and $\boldsymbol{i}-\boldsymbol{j}$, $\boldsymbol{i}+\boldsymbol{k}$.

Let $\boldsymbol{n}_1 = \boldsymbol{i}\times(\boldsymbol{i}+\boldsymbol{j}) = \boldsymbol{i}\times\boldsymbol{j} = \boldsymbol{k}$ and

$$\boldsymbol{n}_2 = (\boldsymbol{i}-\boldsymbol{j})\times(\boldsymbol{i}+\boldsymbol{k})$$
$$= -\boldsymbol{j}+\boldsymbol{k}-\boldsymbol{i} = -\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}.$$

$\therefore$ $\boldsymbol{n}_1$ is perpendicular to l and $\boldsymbol{n}_2$ is also perpendicular to l.

$\therefore$ Since $\boldsymbol{a}$ is parallel to the line l, follows that $\boldsymbol{a}$ is perpendicular to both $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$.

$\therefore$ $\boldsymbol{a}$ is parallel to $\boldsymbol{n}_1\times\boldsymbol{n}_2 = \boldsymbol{k}\times(-\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}) = -\boldsymbol{j}+\boldsymbol{i} = \boldsymbol{i}-\boldsymbol{j}$.

$\therefore$ $\boldsymbol{a} = \lambda(\boldsymbol{n}_1\times\boldsymbol{n}_2) = \lambda(\boldsymbol{i}-\boldsymbol{j})$, for some real λ. Let $\boldsymbol{b} = \boldsymbol{i}-2\boldsymbol{j}+2\boldsymbol{k}$.

Let θ be the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\therefore \cos\theta = \frac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{\lambda(1+2)}{|\lambda|\sqrt{2}\,(3)} = \pm\frac{1}{\sqrt{2}}.$$

$$\therefore \theta = 45^0 \text{ or } 135^0.$$

12. Problem: *Let* $\boldsymbol{a} = 4\boldsymbol{i}+5\boldsymbol{j}-\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i}-4\boldsymbol{j}+5\boldsymbol{k}$ *and* $\boldsymbol{c} = 3\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}$. *Find vector* $\boldsymbol{\alpha}$ *which is perpendicular to both* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *and* $\boldsymbol{\alpha}.\boldsymbol{c} = 21$.

Solution: Since $\boldsymbol{\alpha}$ is perpendicular to both $\boldsymbol{a}$ *and* $\boldsymbol{b}$, there exists scalar λ such that

$$\boldsymbol{\alpha} = \lambda(\boldsymbol{a}\times\boldsymbol{b}) = \lambda\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 4 & 5 & -1\\ 1 & -4 & 5\end{vmatrix}$$
$$= \lambda(21\boldsymbol{i}-21\boldsymbol{j}-21\boldsymbol{k})$$
$$= 21\lambda(\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}).$$
$$\boldsymbol{\alpha}.\boldsymbol{c} = 21 \Rightarrow 21\lambda(3-1+1) = 21$$
$$\therefore \lambda = \frac{1}{3} \text{ and } \boldsymbol{\alpha} = 7\boldsymbol{i}-7\boldsymbol{j}-7\boldsymbol{k}.$$

13. Problem: *For any vector* $\boldsymbol{a}$, *show that* $|\boldsymbol{a}\times\boldsymbol{i}|^2+|\boldsymbol{a}\times\boldsymbol{j}|^2+|\boldsymbol{a}\times\boldsymbol{k}|^2 = 2|\boldsymbol{a}|^2$.

Solution: Let $\boldsymbol{a} = x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$.

Then $\boldsymbol{a}\times\boldsymbol{i} = -y\boldsymbol{k}+z\boldsymbol{j}$.

$$\therefore |\boldsymbol{a}\times\boldsymbol{i}|^2 = y^2+z^2$$

Similarly $|\boldsymbol{a}\times\boldsymbol{j}|^2 = z^2+x^2$ and $|\boldsymbol{a}\times\boldsymbol{k}|^2 = x^2+y^2$

$$\therefore |\boldsymbol{a}\times\boldsymbol{i}|^2+|\boldsymbol{a}\times\boldsymbol{j}|^2+|\boldsymbol{a}\times\boldsymbol{k}|^2 = 2(x^2+y^2+z^2) = 2|\boldsymbol{a}|^2.$$

14. Problem: *If $\boldsymbol{a}$ is a non-zero vector and $\boldsymbol{b}, \boldsymbol{c}$ are two vectors such that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{c}$ and $\boldsymbol{a} \,.\, \boldsymbol{b} = \boldsymbol{a} \,.\, \boldsymbol{c}$, then prove that $\boldsymbol{b} = \boldsymbol{c}$.*

Solution: $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{c} \Rightarrow \boldsymbol{a} \times (\boldsymbol{b} - \boldsymbol{c}) = \boldsymbol{0}$

$\Rightarrow$ either $\boldsymbol{b} = \boldsymbol{c}$ or $\boldsymbol{b} - \boldsymbol{c}$ is collinear with $\boldsymbol{a}$

Again $\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{a}.\boldsymbol{c} \Rightarrow \boldsymbol{a}.(\boldsymbol{b} - \boldsymbol{c}) = 0$

$\Rightarrow$ either $\boldsymbol{b} = \boldsymbol{c}$ or $\boldsymbol{b} - \boldsymbol{c}$ is perpendicular to $\boldsymbol{a}$.

$\therefore$ If $\boldsymbol{b} \neq \boldsymbol{c}$, then $\boldsymbol{b} - \boldsymbol{c}$ is parallel to $\boldsymbol{a}$ and is perpendicular to $\boldsymbol{a}$ which is impossible.

$\therefore$ $\boldsymbol{b} = \boldsymbol{c}$.

Exercise 5(b)

I. 1. If $|\boldsymbol{p}| = 2, |\boldsymbol{q}| = 3$ and $(\boldsymbol{p}, \boldsymbol{q}) = \dfrac{\pi}{6}$, then find $|\boldsymbol{p} \times \boldsymbol{q}|^2$.

2. If $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - 3\boldsymbol{j} - 5\boldsymbol{k}$, then find $|\boldsymbol{a} \times \boldsymbol{b}|$.

3. If $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$, then find $(\boldsymbol{a} + \boldsymbol{b}) \times (\boldsymbol{a} - \boldsymbol{b})$.

4. If $4\boldsymbol{i} + \dfrac{2p}{3}\boldsymbol{j} + p\boldsymbol{k}$ is parallel to the vector $\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$, find p.

5. Compute $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} + \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} + \boldsymbol{b})$.

6. If $\boldsymbol{p} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, then find $|\boldsymbol{p} \times \boldsymbol{k}|^2$.

7. Compute $2\boldsymbol{j} \times (3\boldsymbol{i} - 4\boldsymbol{k}) + (\boldsymbol{i} + 2\boldsymbol{j}) \times \boldsymbol{k}$.

8. Find unit vector perpendicular to both $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$.

9. If θ is the angle between the vectors $\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{j} + \boldsymbol{k}$, then find $\sin\theta$.

10. Find the area of the parallelogram having $\boldsymbol{a} = 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = -\boldsymbol{i} + \boldsymbol{k}$ as adjacent sides.

11. Find the area of the parallelogram whose diagonals are $3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}$.

12. Find the area of the triangle having $3\boldsymbol{i} + 4\boldsymbol{j}$ and $-5\boldsymbol{i} + 7\boldsymbol{j}$ as two of its sides.

13. Find unit vector perpendicular to the plane determined by the vectors $\boldsymbol{a} = 4\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - 6\boldsymbol{j} - 3\boldsymbol{k}$.

14. Find the area of the triangle whose vertices are A(1, 2, 3), B(2, 3, 1) and C(3, 1, 2).

II. 1. If $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = 0$, then prove that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{c} \times \boldsymbol{a}$.

2. If $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{b} = -\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, then find $(\boldsymbol{a} \times \boldsymbol{b}).(\boldsymbol{b} \times \boldsymbol{c})$.

3. Find the vector area and the area of the parallelogram having $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ as adjacent sides.

4. If $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} \neq \boldsymbol{0}$, then show that $\boldsymbol{a} + \boldsymbol{c} = p\boldsymbol{b}$, where p is some scalar.

5. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be vectors, satisfying $|\boldsymbol{a}| = |\boldsymbol{b}| = 5$ and $(\boldsymbol{a}, \boldsymbol{b}) = 45^0$. Find the area of the triangle having $\boldsymbol{a} - 2\boldsymbol{b}$ and $3\boldsymbol{a} + 2\boldsymbol{b}$ as two of its sides.

6. Find the vector having magnitude $\sqrt{6}$ units and perpendicular to both $2\boldsymbol{i} - \boldsymbol{k}$ and $3\boldsymbol{j} - \boldsymbol{i} - \boldsymbol{k}$.

7. Find a unit vector perpendiclar to the plane determined by the points P(1, –1, 2), Q(2, 0, –1) and R(0, 2, 1).

8. If $\boldsymbol{a} . \boldsymbol{b} = \boldsymbol{a} . \boldsymbol{c}$ and $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{c}$, $\boldsymbol{a} \neq \boldsymbol{0}$, then show that $\boldsymbol{b} = \boldsymbol{c}$.

9. Find a vector of magnitude 3 and perpendicular to both the vectors $\boldsymbol{b} = 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$.

10. If $|\boldsymbol{a}| = 13$, $|\boldsymbol{b}| = 5$ and $\boldsymbol{a} . \boldsymbol{b} = 60$, then find $|\boldsymbol{a} \times \boldsymbol{b}|$.

11. Find unit vector perpendicular to the plane passing through the points (1, 2, 3), (2, –1, 1) and (1, 2, –4).

III.1. If $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ represent the vertices A, B and C respectively of ΔABC, then prove that $|(\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{b} \times \boldsymbol{c}) + (\boldsymbol{c} \times \boldsymbol{a})|$ is twice the area of ΔABC.

2. If $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$, then compute $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ and verify that it is perpendicualr to $\boldsymbol{a}$.

3. If $\boldsymbol{a} = 7\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} + 8\boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, then compute $\boldsymbol{a} \times \boldsymbol{b}$, $\boldsymbol{a} \times \boldsymbol{c}$ and $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c})$. Verify whether the cross product is distributive over vector addition.

4. If $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{c} = \boldsymbol{j} - \boldsymbol{k}$, then find vector $\boldsymbol{b}$ such that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c}$ and $\boldsymbol{a} . \boldsymbol{b} = 3$.

5. $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are three vectors of equal magnitudes and each of them is inclined at an angle of 60^0 to the others. If $|\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}| = \sqrt{6}$, then find $|\boldsymbol{a}|$.

6. For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, show that

$$\left(1 + |\boldsymbol{a}|^2\right)\left(1 + |\boldsymbol{b}|^2\right) = |1 - \boldsymbol{a}.\boldsymbol{b}|^2 + |\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{b}|^2.$$

7. If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are unit vectors such that $\boldsymbol{a}$ is perpendicular to the plane of $\boldsymbol{b}, \boldsymbol{c}$ and the angle between $\boldsymbol{b}$ and $\boldsymbol{c}$ is $\frac{\pi}{3}$, then find $|\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}|$.

8. $\boldsymbol{a} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{c} = 4\boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{d} = \boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$, then compute the following.
   (i) $(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d})$ and (ii) $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} - (\boldsymbol{a} \times \boldsymbol{d}) \cdot \boldsymbol{b}$.

5.10 Scalar triple product

In this section we introduce the concept of scalar triple product of three vectors and discuss its properties and its geometrical interpretation.

5.10.1 Definition

*Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be three vectors. We call $(\boldsymbol{a} \times \boldsymbol{b}).\boldsymbol{c}$, **the scalar triple product** of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ and denote this by $[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]$. Usually $[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]$ is called box $[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]$.*

5.10.2 Note: $(\boldsymbol{a} \times \boldsymbol{b}).\boldsymbol{c} = 0$ when

(i) one of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ is $\mathbf{0}$ or

(ii) $\boldsymbol{a}, \boldsymbol{b}$ or $\boldsymbol{b}, \boldsymbol{c}$ or $\boldsymbol{c}, \boldsymbol{a}$ are collinear vectors or

(iii) $\boldsymbol{c}$ is perpendicular to $\boldsymbol{a} \times \boldsymbol{b}$.

5.10.3 Theorem: *Let $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ be three non-coplanar vectors and* **OA** $= \boldsymbol{a}$, **OB** $= \boldsymbol{b}$ *and* **OC** $= \boldsymbol{c}$. *Let V be the volume of the parallelopiped with* **OA, OB** *and* **OC** *as coterminus edges. Then*

(i) $(\boldsymbol{a} \times \boldsymbol{b}).\boldsymbol{c} = V$, if $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is a right handed system.

(ii) $(\boldsymbol{a} \times \boldsymbol{b}).\boldsymbol{c} = -V$, if $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is a left handed system.

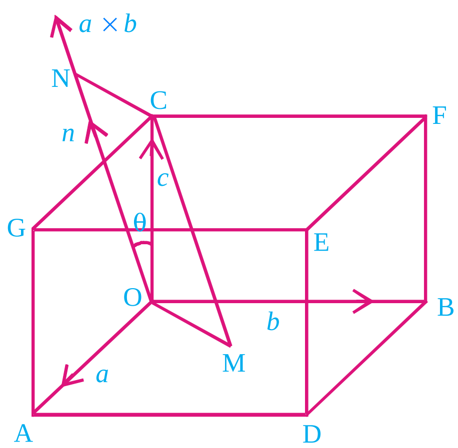


Fig. 5.22

Proof : (i) Consider the parallelopiped OADBFCGE having **OA, OB** and **OC** as coterminus edges. Assume that $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ is right handed system. Draw CM perpendicular to the plane determined by **OA** and **OB** (i.e., $\boldsymbol{a}$ and $\boldsymbol{b}$) and N be the foot of the perpendicular to the support of $\boldsymbol{a} \times \boldsymbol{b}$ (see Fig. 5.22). Let $\boldsymbol{n}$ be the unit vector in the direction of $\boldsymbol{a} \times \boldsymbol{b}$ so that by definition of $\boldsymbol{a} \times \boldsymbol{b}$, we have $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{n})$ is a right handed system. Let θ be the angle between $\boldsymbol{a} \times \boldsymbol{b}$ and $\boldsymbol{c}$. i.e., $\theta = \angle CON$.

V = (Area of the base parallelogram OADB) $\times$ (height of the vertex C from the base)

$$= |\boldsymbol{a}\times\boldsymbol{b}|(\text{CM}) = |\boldsymbol{a}\times\boldsymbol{b}|\,(\text{ON}). \qquad \ldots (1)$$

But from ΔOCN, ON = (OC) cos θ

$\therefore$ $\quad V = |\boldsymbol{a}\times\boldsymbol{b}|\,(\text{OC})\cos\theta \quad$ (from (1))

$= |\boldsymbol{a}\times\boldsymbol{b}|\,|\boldsymbol{c}|\cos\theta$

$= (\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c}$.

Thus V = $(\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c} = [\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]$.

(ii) Suppose $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is a left handed system.

$\therefore (\boldsymbol{a}, \boldsymbol{b}, -\boldsymbol{c})$ is a right handed system (see note 5.7.2). But the volumes of the corresponding parallelopipeds are same.

$\therefore$ V = $(\boldsymbol{a}\times\boldsymbol{b}).(-\boldsymbol{c}) = -[(\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c}]$

$\therefore (\boldsymbol{a}\times\boldsymbol{b}).\,\boldsymbol{c} = -V$.

5.10.4 Theorem: *For any three vectors* **a, b** *and* **c**

$(\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c} = (\boldsymbol{b}\times\boldsymbol{c}).\boldsymbol{a} = (\boldsymbol{c}\times\boldsymbol{a}).\,\boldsymbol{b}$ *that is,* $[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}] = [\boldsymbol{b}\ \boldsymbol{c}\ \boldsymbol{a}] = [\boldsymbol{c}\ \boldsymbol{a}\ \boldsymbol{b}]$.

Proof : If one of **a, b** and **c** is **O** or any two are collinear, the equality holds (by 5.10.2).

Assume that $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$, $(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{a})$ and $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b})$ form right handed systems.

$\therefore (\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c} = (\boldsymbol{b}\times\boldsymbol{c}).\boldsymbol{a} = (\boldsymbol{c}\times\boldsymbol{a}).\boldsymbol{b}$ = volume of the parallelopiped = V.

If all the triads form left handed systems, then

$(\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c} = (\boldsymbol{b}\times\boldsymbol{c}).\boldsymbol{a} = (\boldsymbol{c}\times\boldsymbol{a}).\boldsymbol{b} = -$ V

Thus $(\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c} = (\boldsymbol{b}\times\boldsymbol{c}).\boldsymbol{a} = (\boldsymbol{c}\times\boldsymbol{a}).\boldsymbol{b}$

5.10.5 Theorem : *If* **a, b, c** *are any three vectors, then* $(\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c} = \boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c})$. *(that is, in a scalar triple product, the operations dot and cross can be interchanged)*

Proof : From Theorem 5.10.4, we have

$(\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c} = (\boldsymbol{b}\times\boldsymbol{c}).\boldsymbol{a} = \boldsymbol{a}\,.(\boldsymbol{b}\times\boldsymbol{c})$ ($\because$ dot product is commutative)

5.10.6 Theorem: *If* **a, b, c** *are three nonzero vectors such that no two are collinear, then* $[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}] = 0$ *if and only if* **a, b** *and* **c** *are coplanar.*

Proof: *Suppose* **a, b** *and* **c** *are coplanar. Since* $\boldsymbol{a}\times\boldsymbol{b}$ *is perpendicular to the plane determined by* **a** *and* **b** *it is also perpendicular to* **c**. *Hence* $(\boldsymbol{a}\times\boldsymbol{b}).\,\boldsymbol{c} = 0$.

$$\therefore\ [\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}] = 0.$$

Conversely assume that $[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}] = 0$ i.e, $(\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c} = 0$.

$\therefore \boldsymbol{a}\times\boldsymbol{b}$ is perpendicular to **c**. But $\boldsymbol{a}\times\boldsymbol{b}$ is perpendicular to both **a** and **b**.

$\therefore \boldsymbol{a}\times\boldsymbol{b}$ is perpendicular to **a, b** and **c** .

$\therefore \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are coplanar.

5.10.7 Corollary

Four distinct points A, B, C *and* D *are coplanar if and only if* $[\mathbf{AB}\ \mathbf{AC}\ \mathbf{AD}] = 0$.

Proof : A, B, C and D are coplanar $\Leftrightarrow$ the vectors **AB, AC** and **AD** are coplanar

$\Leftrightarrow [\mathbf{AB}\ \mathbf{AC}\ \mathbf{AD}] = 0$.

5.10.8 Theorem: *Let $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ be orthogonal triad of unit vectors which is a right handed system. Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.*

Then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Proof: It is known that $\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$

$$\therefore [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = (\mathbf{a} \times \mathbf{b}).\mathbf{c}$$

$$= (a_2 b_3 - a_3 b_2)c_1 - (a_1 b_3 - a_3 b_1)c_2 + (a_1 b_2 - a_2 b_1)c_3$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

5.10.9 Corollary

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. *Then* **a, b, c** *are coplanar if and only if* $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$.

Proof: Follows from Theroems 5.10.6 and 5.10.8.

5.10.10 Corollary

Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ *be three noncoplanar vectors and* $\mathbf{a} = a_1\boldsymbol{\alpha} + a_2\boldsymbol{\beta} + a_3\boldsymbol{\gamma}$, $\mathbf{b} = b_1\boldsymbol{\alpha} + b_2\boldsymbol{\beta} + b_3\boldsymbol{\gamma}$, $\mathbf{c} = c_1\boldsymbol{\alpha} + c_2\boldsymbol{\beta} + c_3\boldsymbol{\gamma}$. *Then* **a, b** *and* **c** *are coplanar if and only if*

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

Proof : From Theorem 5.9.8, $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \boldsymbol{\beta}\times\boldsymbol{\gamma} & \boldsymbol{\gamma}\times\boldsymbol{\alpha} & \boldsymbol{\alpha}\times\boldsymbol{\beta} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$= (a_2 b_3 - a_3 b_2)(\boldsymbol{\beta}\times\boldsymbol{\gamma}) - (a_1 b_3 - a_3 b_1)(\boldsymbol{\gamma}\times\boldsymbol{\alpha}) + (a_1 b_2 - a_2 b_1)(\boldsymbol{\alpha}\times\boldsymbol{\beta})$$

$$\therefore [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = (\mathbf{a} \times \mathbf{b}).\mathbf{c}$$

$$= (a_2 b_3 - a_3 b_2)(\boldsymbol{\beta}\times\boldsymbol{\gamma}).(c_1\boldsymbol{\alpha}) - (a_1 b_3 - a_3 b_1)(\boldsymbol{\gamma}\times\boldsymbol{\alpha}).(c_2\boldsymbol{\beta})$$

$$+ (a_1 b_2 - a_2 b_1)(\boldsymbol{\alpha}\times\boldsymbol{\beta}).(c_3\boldsymbol{\gamma})$$

$$= \left[(a_2 b_3 - a_3 b_2) c_1 - (a_1 b_3 - a_3 b_1) c_2 + (a_1 b_2 - a_2 b_1) c_3\right] [\boldsymbol{\alpha}\ \boldsymbol{\beta}\ \boldsymbol{\gamma}],$$

$$(\because [\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\gamma}] = [\boldsymbol{\beta}\boldsymbol{\gamma}\boldsymbol{\alpha}] = [\boldsymbol{\gamma}\boldsymbol{\alpha}\boldsymbol{\beta}])$$

$$\therefore [\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\boldsymbol{\alpha}\ \boldsymbol{\beta}\ \boldsymbol{\gamma}].$$

Since $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are non-coplanar, $[\boldsymbol{\alpha}\ \boldsymbol{\beta}\ \boldsymbol{\gamma}] \neq 0$.

$\therefore$ $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are coplanar vectors if and only if $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$.

5.10.11 Theorem: *The volume of a tetrahedron with* $\boldsymbol{a}, \boldsymbol{b}$ *and* $\boldsymbol{c}$ *as coterminus edges is* $\frac{1}{6}\left|[\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}]\right|$.

Proof: Let OABC be a tetrahedron and $\mathbf{OA} = \boldsymbol{a}$, $\mathbf{OB} = \boldsymbol{b}$, $\mathbf{OC} = \boldsymbol{c}$ (Fig. 5.23). Let V be the volume of the tetrahedron OABC. By definition, the volume V is given by

$$V = \frac{1}{3} \text{ (area of the base } \Delta\text{OAB) (length of the perpendicular from C to the base } \Delta\text{OAB}).$$

CN is the perpendicular from C to ΔOAB and CM is the perpendicular from C onto the supporting line of $\boldsymbol{a} \times \boldsymbol{b}$ so that CN = OM = Length of the projection of $\boldsymbol{c}$ onto $\boldsymbol{a} \times \boldsymbol{b}$

$$= \frac{|(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}|}{|\boldsymbol{a} \times \boldsymbol{b}|} = \frac{|[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]|}{|\boldsymbol{a} \times \boldsymbol{b}|}$$

$$\text{Area of } \Delta\text{OAB} = \frac{|\boldsymbol{a} \times \boldsymbol{b}|}{2}$$

$$\therefore V = \frac{1}{3} \times \frac{1}{2} |\boldsymbol{a} \times \boldsymbol{b}| \frac{|[\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}]|}{|\boldsymbol{a} \times \boldsymbol{b}|}$$

$$= \frac{1}{6} |[\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}]| .$$

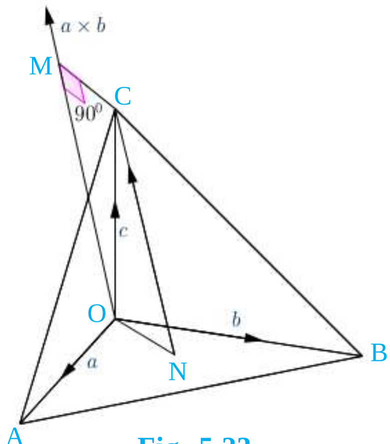


Fig. 5.23

5.10.12 Corollary

The volume of the tetrahedron whose vertices are A, B, C *and* D *is* $\frac{1}{6}\left|[\mathbf{DA}\ \mathbf{DB}\ \mathbf{DC}]\right|$.

Proof: Since **DA, DB** and **DC** are coterminus edges of the tetrahedron ABCD, from the above theorem, it follows that its volume is $\frac{1}{6}\left|[\mathbf{DA}\ \mathbf{DB}\ \mathbf{DC}]\right|$.

5.11 Vector equation of a plane - different forms, skew lines, shortest distance -plane, condition for coplanarity etc.

In Chapter 4, we have discussed about the parametric vectorial equation of a plane in various forms. In this section we obtain the vector equations of a plane by using dot and cross products. Also, we introduce the concept of skew lines, the shortest distance between two skew lines and derive a formula for the shortest distance. In this connection, we fix the origin of reference 'O'. $\boldsymbol{a}$ is a point means $\boldsymbol{a}$ is the position vector of a point with respect to origin 'O'.

5.11.1 Theorem: *The vector equation of a plane passing through the point* A $(\boldsymbol{a})$ *and parallel to two non-collinear vectors* $\boldsymbol{b}$ *and* $\boldsymbol{c}$ *is* $[\boldsymbol{r}\,\boldsymbol{b}\,\boldsymbol{c}] = [\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}]$.

Proof: Let $\boldsymbol{a}$ represent the point A and P$(\boldsymbol{r})$ be any point in the plane. We may assume that $A \neq P$.

P lies in the plane

$\Rightarrow$ The vectors **AP**, $\boldsymbol{b}, \boldsymbol{c}$ are coplanar

$\Rightarrow$ $[\mathbf{AP}\ \boldsymbol{b}\ \boldsymbol{c}] = 0$ (by Theorem 5.10.6)

$\Rightarrow$ $\mathbf{AP}.(\boldsymbol{b} \times c) = 0$

$\Rightarrow$ $(\boldsymbol{r} - \boldsymbol{a}).(\boldsymbol{b} \times \boldsymbol{c}) = 0$

$\Rightarrow$ $\boldsymbol{r}.(\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c})$

$\Rightarrow$ $[\boldsymbol{r}\,\boldsymbol{b}\,\boldsymbol{c}] = [\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}]$.

Suppose P$(\boldsymbol{r})$ is any point in the space such that $[\boldsymbol{r}\ \boldsymbol{b}\ \boldsymbol{c}] = [\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]$.

In the above argument, if we retrace the steps backwards, we will have $[\mathbf{AP}\ \boldsymbol{b}\ \boldsymbol{c}] = 0$.

Thus the vectors **AP**, $\boldsymbol{b}, \boldsymbol{c}$ are coplanar. Hence P lies in the plane.

5.11.2 Theorem: *The vector equation of the plane passing through points* A$(\boldsymbol{a})$, B$(\boldsymbol{b})$ *and parallel to the vector* $\boldsymbol{c}$ *is* $[\boldsymbol{r}\ \boldsymbol{b}\ \boldsymbol{c}] + [\boldsymbol{r}\ \boldsymbol{c}\ \boldsymbol{a}] = [\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}]$.

Proof: Let P$(\boldsymbol{r})$ be any point. We may assume that $P \neq A$.

Then P lies in the plane $\Leftrightarrow$ the vector $\mathbf{AP} \times \mathbf{AB}$ is perpendicular to the plane

$\Leftrightarrow$ $\mathbf{AP} \times \mathbf{AB}$ is perpendicular to the vector $\boldsymbol{c}$.

$\Leftrightarrow$ $(\mathbf{AP} \times \mathbf{AB}).\boldsymbol{c} = 0$.

$\Leftrightarrow$ $\mathbf{AP}.(\mathbf{AB} \times \boldsymbol{c}) = 0$ (Theorem 5.10.5)

$\Leftrightarrow$ $(\boldsymbol{r} - \boldsymbol{a}).((\boldsymbol{b} - \boldsymbol{a}) \times \boldsymbol{c}) = 0$

$\Leftrightarrow$ $(\boldsymbol{r} - \boldsymbol{a}).(\boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{c} \times \boldsymbol{a}) = 0$

$\Leftrightarrow$ $[\boldsymbol{r}\ \boldsymbol{b}\ \boldsymbol{c}] + [\boldsymbol{r}\ \boldsymbol{c}\ \boldsymbol{a}] = [\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}]$.

5.11.3 Theorem: *The vector equation of the plane passing through three non-collinear points* A$(\boldsymbol{a})$, B$(\boldsymbol{b})$ and C$(\boldsymbol{c})$ *is* $[\boldsymbol{r}\,\boldsymbol{b}\,\boldsymbol{c}] + [\boldsymbol{r}\,\boldsymbol{c}\,\boldsymbol{a}] + [\boldsymbol{r}\,\boldsymbol{a}\,\boldsymbol{b}] = [\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}]$.

Proof: Let P$(\boldsymbol{r})$ be any point.

The four points A, B, C and P are coplanar

$\Leftrightarrow$ The vectors **AP, AB** and **AC** are coplanar

$\Leftrightarrow$ $\boldsymbol{r}-\boldsymbol{a}$, $\boldsymbol{b}-\boldsymbol{a}$ and $\boldsymbol{c}-\boldsymbol{a}$ are coplanar.

$$\Leftrightarrow \left[\boldsymbol{r}-\boldsymbol{a}\;\; \boldsymbol{b}-\boldsymbol{a}\;\; \boldsymbol{c}-\boldsymbol{a}\right]=0$$

$$\Leftrightarrow (\boldsymbol{r}-\boldsymbol{a}).(\boldsymbol{b}-\boldsymbol{a})\times(\boldsymbol{c}-\boldsymbol{a})=0$$

$$\Leftrightarrow (\boldsymbol{r}-\boldsymbol{a}).\{\boldsymbol{b}\times\boldsymbol{c}+\boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{c}\times\boldsymbol{a}\}=0$$

$$\Leftrightarrow \boldsymbol{r}.[\boldsymbol{b}\times\boldsymbol{c}+\boldsymbol{c}\times\boldsymbol{a}+\boldsymbol{a}\times\boldsymbol{b}]=[\boldsymbol{a}\;\boldsymbol{b}\;\boldsymbol{c}]$$

$$\Leftrightarrow [\boldsymbol{r}\;\boldsymbol{b}\;\boldsymbol{c}]+[\boldsymbol{r}\;\boldsymbol{c}\;\boldsymbol{a}]+[\boldsymbol{r}\;\boldsymbol{a}\;\boldsymbol{b}]=[\boldsymbol{a}\;\boldsymbol{b}\;\boldsymbol{c}].$$

5.11.4 Theorem: *The vector equation of the plane containing the line* $\boldsymbol{r}=\boldsymbol{a}+t\boldsymbol{b}$, $t\in\mathbf{R}$ *and perpendicular to the plane* $\boldsymbol{r}.\boldsymbol{c}=q$ *is* $[\boldsymbol{r}\;\boldsymbol{b}\;\boldsymbol{c}]=[\boldsymbol{a}\;\boldsymbol{b}\;\boldsymbol{c}]$.

Proof: For the plane $\boldsymbol{r}.\boldsymbol{c}=\boldsymbol{q}$, the vector $\boldsymbol{c}$ is a normal. Since the plane contains the line $\boldsymbol{r}=\boldsymbol{a}+t\boldsymbol{b}$, it passes through the point $\boldsymbol{a}$ and is parallel to the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$.

$\therefore$ By Theorem 5.11.1, the vector equation of the plane is $[\boldsymbol{r}\;\boldsymbol{b}\;\boldsymbol{c}]=[\boldsymbol{a}\;\boldsymbol{b}\;\boldsymbol{c}]$.

5.11.5 Skew lines, Shortest distance and Cartesian equivalents.

If two lines in space intersect at a point, then the shortest distance between them is zero. Also, if two lines in space are parallel, then the shortest distance between them is the perpendicular distance or the length of the perpendicular drawn from any point on one of the lines onto the other line.

In a space, there are pairs of lines which are neither intersecting nor parallel. Such a pair of lines is called a pair of skew lines. Thus, two lines are called skew lines, if there is no plane containing both the lines.

5.11.6 Example

Consider a room of size 1, 3, 2 units along X, Y and Z axes respectively. (Fig. 5.24).

The line GE that goes diagonally across the ceiling and the line DB passing through one corner of the ceiling directly above A, goes diagonally down the wall. These lines are skew lines because they are not parallel and also never meet.

By the shortest distance between two lines we mean the join of a point on one line with a point on the other line so that the length of the segment so obtained is the smallest. In the case of skew lines, the line of the shortest distance will be perpendicular to both the lines.

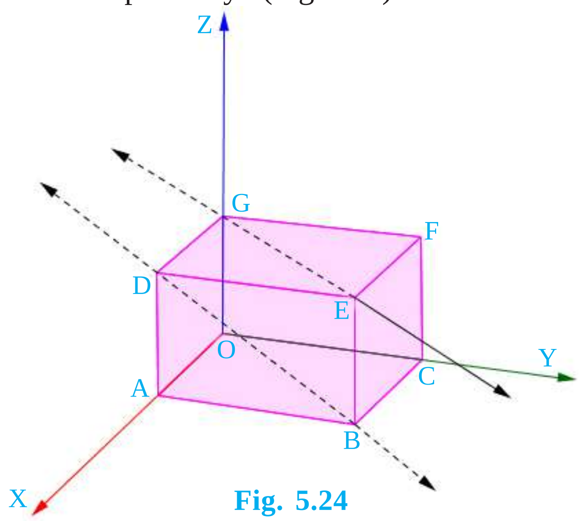


Fig. 5.24

5.11.7 Distance between two skew lines

Let L_1 and L_2 be two skew lines, as shown in Fig. 5.25, with equations.

$$\boldsymbol{r} = \boldsymbol{a}_1 + \lambda \boldsymbol{b}_1.$$

and $$\boldsymbol{r} = \boldsymbol{a}_2 + \mu \boldsymbol{b}_2.$$

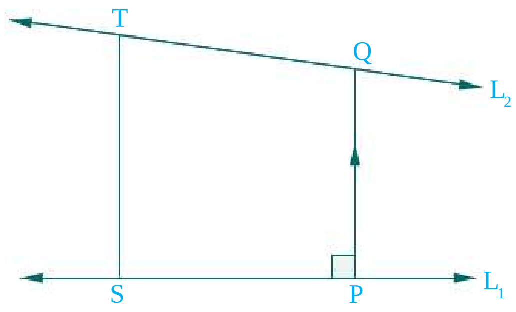


Fig. 5.25

Let S be the point on L_1 with position vector $\boldsymbol{a}_1$ and let T be the point on L_2 with position vector $\boldsymbol{a}_2$. Then the magnitude of the vector of shortest distance will be equal to that of the projection of ST along the direction of the line of shortest distance.

If **PQ** is the vector of shortest distance between L_1 and L_2, then it is perpendicular to both $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$. The unit vector $\boldsymbol{n}$ along **PQ** would therefore be $\boldsymbol{n} = \dfrac{\boldsymbol{b}_1 \times \boldsymbol{b}_2}{|\boldsymbol{b}_1 \times \boldsymbol{b}_2|}$. Then $\mathbf{PQ} = d\,\boldsymbol{n}$, where d is the magnitude of the shortest distance vector.

Let θ be the angle between **ST** and **PQ**. Then

$$PQ = ST\ .\ |\cos\theta|$$

But $$\cos\theta = \left|\frac{\mathbf{PQ}\ .\ \mathbf{ST}}{|\mathbf{PQ}||\mathbf{ST}|}\right|$$

$$= \left|\frac{d\,\boldsymbol{n}.(\boldsymbol{a}_2 - \boldsymbol{a}_1)}{d\ \text{ST}}\right|, \text{ since } \mathbf{ST} = \boldsymbol{a}_2 - \boldsymbol{a}_1.$$

$$= \left|\frac{(\boldsymbol{b}_1 \times \boldsymbol{b}_2)\ .\ (\boldsymbol{a}_2 - \boldsymbol{a}_1)}{\text{ST}\ |\boldsymbol{b}_1 \times \boldsymbol{b}_2|}\right|, \text{ since } \boldsymbol{n} = \frac{\boldsymbol{b}_1 \times \boldsymbol{b}_2}{|\boldsymbol{b}_1 \times \boldsymbol{b}_2|}$$

Hence the required shortest distance is

$$d = PQ = ST|\cos\theta| = \left|\frac{(\boldsymbol{b}_1 \times \boldsymbol{b}_2)\ .\ (\boldsymbol{a}_2 - \boldsymbol{a}_1)}{|\boldsymbol{b}_1 \times \boldsymbol{b}_2|}\right|.$$

5.11.8 Cartesian form

The shortest distance between the lines

$$l_1 : \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} \quad \text{and} \quad l_2 : \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

is $$\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}.$$

5.11.9 Plane passing through the intersection of two given planes

Let Π_1 and Π_2 be two given planes given by the equations $\boldsymbol{r} \cdot \boldsymbol{n}_1 = d_1$ and $\boldsymbol{r} \cdot \boldsymbol{n}_2 = d_2$ respectively. The position vector of any point on the line of intersection must satisfy both the equations (Fig. 5.26).

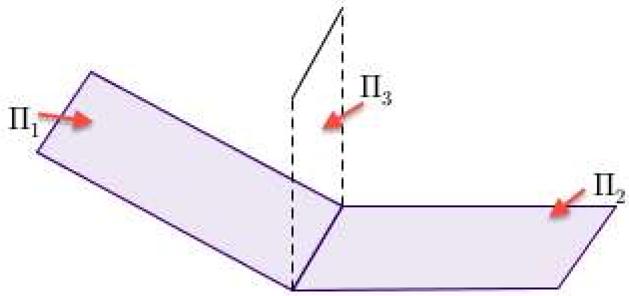


Fig. 5.26

If $\boldsymbol{t}$ is the position vector of a point on the line, then

$\boldsymbol{t} \cdot \boldsymbol{n}_1 = d_1$ and $\boldsymbol{t} \cdot \boldsymbol{n}_2 = d_2$.

Therefore for all real values of λ, we have

$\boldsymbol{t} \cdot (\boldsymbol{n}_1 + \lambda \boldsymbol{n}_2) = d_1 + \lambda d_2$.

Since $\boldsymbol{t}$ is arbitrary, it satisfies for any point on the line. Hence, $\boldsymbol{r} \cdot (\boldsymbol{n}_1 + \lambda \boldsymbol{n}_2) = d_1 + \lambda d_2$ represents a plane Π_3 which is such that if any vector $\boldsymbol{r}$ satisfies both the equations Π_1 and Π_2, it also satisfies the equation of Π_3 i.e., any plane passing through the intersection of the planes $\boldsymbol{r} \cdot \boldsymbol{n}_1 = d_1$ and $\boldsymbol{r} \cdot \boldsymbol{n}_2 = d_2$ has the equation $\boldsymbol{r} \cdot (\boldsymbol{n}_1 + \lambda \boldsymbol{n}_2) = d_1 + \lambda d_2$.

5.11.10 Cartesian form

In the Cartesian system, let

$$\boldsymbol{n}_1 = a_1\boldsymbol{i} + b_1\boldsymbol{j} + c_1\boldsymbol{k}, \quad \boldsymbol{n}_2 = a_2\boldsymbol{i} + b_2\boldsymbol{j} + c_2\boldsymbol{k}, \quad \boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}.$$

Then $x(a_1 + \lambda a_2) + y(b_1 + \lambda b_2) + z(c_1 + \lambda c_2) = \boldsymbol{r} \cdot (\boldsymbol{n}_1 + \lambda \boldsymbol{n}_2) = d_1 + \lambda d_2$.

or $(a_1x + b_1y + c_1z - d_1) + \lambda(a_2x + b_2y + c_2z - d_2) = 0$

is the required cartesian form of the equation of the plane passing through the intersection of the given planes, λ being the parameter.

5.11.11 Condition for coplanarity of two lines

Let the given lines be

$$\boldsymbol{r} = \boldsymbol{a}_1 + \lambda \boldsymbol{b}_1 \qquad \text{... (1)}$$

and

$$\boldsymbol{r} = \boldsymbol{a}_2 + \mu \boldsymbol{b}_2 \qquad \text{... (2)}$$

If line (1) passes through the point A with position vector $\boldsymbol{a}_1$ and is parallel to $\boldsymbol{b}_1$ and line (2) passes through the point B with position vector $\boldsymbol{a}_2$ and is parallel to $\boldsymbol{b}_2$, then $\mathbf{AB} = \boldsymbol{a}_2 - \boldsymbol{a}_1$.

The given lines are coplanar if and only if $\mathbf{AB}$ is perpendicular to $\boldsymbol{b}_1 \times \boldsymbol{b}_2$.

i.e., $\mathbf{AB} \cdot (\boldsymbol{b}_1 \times \boldsymbol{b}_2) = 0$ or $(\boldsymbol{a}_2 - \boldsymbol{a}_1) \cdot (\boldsymbol{b}_1 \times \boldsymbol{b}_2) = 0$.

Cartesian form

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the coordinates of the points A and B respectively. Let (a_1, b_1, c_1) be the direction ratios of $\boldsymbol{b}_1$ and (a_2, b_2, c_2) be the direction ratios of $\boldsymbol{b}_2$. Then

$$\mathbf{AB} = (x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j} + (z_2 - z_1)\boldsymbol{k}$$

$$\boldsymbol{b}_1 = a_1\boldsymbol{i} + b_1\boldsymbol{j} + c_1\boldsymbol{k} \text{ and } \boldsymbol{b}_2 = a_2\boldsymbol{i} + b_2\boldsymbol{j} + c_2\boldsymbol{k}$$

The given lines are coplanar if and only if $\mathbf{AB} \cdot (\boldsymbol{b}_1 \times \boldsymbol{b}_2) = 0$. In the cartesian form, it can be expressed as

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

5.11.12 Perpendicular distance of a Point from a plane

Vector form : Consider a point P with Position vector $\boldsymbol{a}$ and a plane Π_1 whose equation is $\boldsymbol{r}.\boldsymbol{n} = d$. (Fig. 5.27(a), (b)).

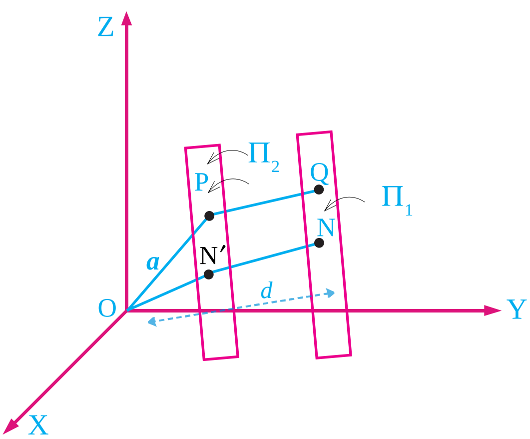


Fig. 5.27(a)

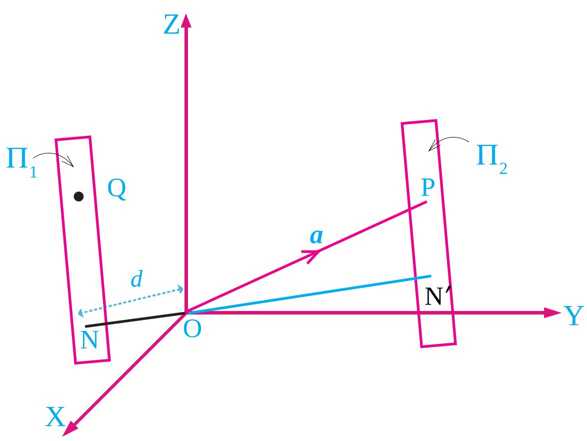


Fig. 5.27(b)

Consider a plane Π_2 through P parallel to the plane Π_1. Thus $\boldsymbol{n}$ is also a unit vector normal to Π_2. Hence its equation is $(\boldsymbol{r} - \boldsymbol{a}) \cdot \boldsymbol{n} = 0$ or $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$.

Let Q be the foot of the perpendicular from P to Π_1, $\mathbf{N'}$ be the foot of the perpendicular from the origin to Π_2 and N be the foot of the perpendicular from $\mathbf{N'}$ to Π_1. Then O, $\mathbf{N'}$, $\mathbf{N}$ are collinear.

Thus the distance ON′ of this plane from the origin is $|\boldsymbol{a} \cdot \boldsymbol{n}|$. The distance of P from the plane Π_1 (Fig.5.27(a)) is PQ, ON – ON′ = $|d - \boldsymbol{a}.\boldsymbol{n}|$ which is the length of the perpendicular from a point $\boldsymbol{a}$ to the given plane. We may establish similar results for Fig. 5.27(b).

5.11.13 Note : (i) If the equation of a plane Π is in the form $\boldsymbol{r} \cdot \mathbf{N} = d$, where $\mathbf{N}$ is normal to the plane, then the perpendicular distance of this plane from a point $\boldsymbol{a}$ is $\dfrac{|\boldsymbol{a}.\mathbf{N} - d|}{|\mathbf{N}|}$.

(ii) The length of the perpendicular from origin O to the plane $\boldsymbol{r} \cdot \mathbf{N} = d$ is $\dfrac{|d|}{|\mathbf{N}|}$, since $\boldsymbol{a} = \mathbf{0}$.

5.11.14 Cartesian form

Let $P(x_1, y_1, z_1)$ be the given point with position vector $\boldsymbol{a}$ and $Ax + By + Cz = D$ be the cartesian equation of the given plane. Then

$$\boldsymbol{a} = x_1 \boldsymbol{i} + y_1 \boldsymbol{j} + z_1 \boldsymbol{k}$$

$$\mathbf{N} = A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}$$

Hence, from 5.11.13(i), the perpendicular distance from P to the plane is

$$\left| \frac{(x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}) \cdot (A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}) - D}{\sqrt{A^2 + B^2 + C^2}} \right| = \left| \frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}} \right|.$$

5.11.15 Angle between a line and a plane

Vector form : The angle between a line and a plane is the complement of the angle between the line and normal to the plane (Fig. 5.28).

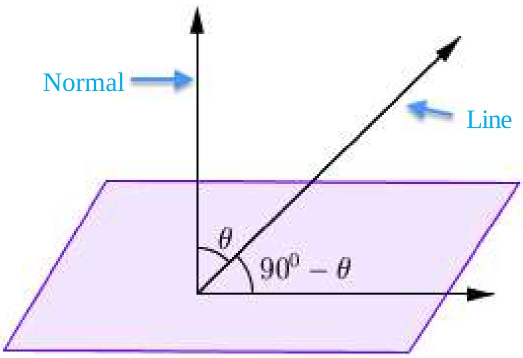


Fig. 5.28

If the equation of the line is $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$ and the equation of the plane is $\boldsymbol{r} \cdot \boldsymbol{n} = d$, then the angle θ between the line and the normal to the plane is $\cos\theta = \left| \dfrac{\boldsymbol{b} \cdot \boldsymbol{n}}{|\boldsymbol{b}||\boldsymbol{n}|} \right|$.

Hence the angle ϕ between the line and the plane is given by $(90^0 - \theta)$.

$$\therefore \sin\phi = \sin(90^0 - \theta) = \cos\theta = \left| \frac{\boldsymbol{b} \cdot \boldsymbol{n}}{|\boldsymbol{b}||\boldsymbol{n}|} \right| \quad \text{or} \quad \phi = \sin^{-1}\left| \frac{\boldsymbol{b} \cdot \boldsymbol{n}}{|\boldsymbol{b}||\boldsymbol{n}|} \right|.$$

5.12 Vector triple product - results

In section 5.10, we have introduced the concept of scalar triple product of three vectors and in section 5.11 studied some of its properties and its applications in deriving the equation of a plane in different forms. Suppose ***a, b, c*** are three vectors. Then $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}$ is called the vector triple product or vector product of three vectors. In this section we study some properties of the vector product $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}$ of three vectors ***a, b, c***.

5.12.1 Theorem: *Let* ***a, b, c*** *be three vectors. Then*

(i) $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=(\boldsymbol{a}.\boldsymbol{c})\,\boldsymbol{b}-(\boldsymbol{b}.\,\boldsymbol{c})\,\boldsymbol{a}$

(ii) $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}.\boldsymbol{c})\,\boldsymbol{b}-(\boldsymbol{a}.\,\boldsymbol{b})\,\boldsymbol{c}$

Proof

(i) Without loss of generality, we may assume that ***a*** and ***b*** are non-collinear vectors and ***c*** is not parallel to $\boldsymbol{a}\times\boldsymbol{b}$, as otherwise $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=\boldsymbol{0}=(\boldsymbol{a}.\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{b}.\boldsymbol{c})\boldsymbol{a}$. Fix the origin 'O'. Let **OA** = ***a***, **OB** = ***b***. We consider the plane OAB as XY-plane. Let ***i*** be the unit vector in the direction of **OA** and ***j*** be unit vector perpendicular to ***i*** in the XY-plane. Fix the unit vector ***k*** perpendicular to *xy*-plane such that $(\boldsymbol{i},\boldsymbol{j},\boldsymbol{k})$ is an orthogonal triad of unit vectors forming a right handed system. Then, we can write $\boldsymbol{a}=a_1\boldsymbol{i}$, $\boldsymbol{b}=b_1\boldsymbol{i}+b_2\boldsymbol{j}$ and $\boldsymbol{c}=c_1\boldsymbol{i}+c_2\boldsymbol{j}+c_3\boldsymbol{k}$.

$$\therefore (\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=(a_1b_2\boldsymbol{k})\times(c_1\boldsymbol{i}+c_2\boldsymbol{j}+c_3\boldsymbol{k})$$

$$=(a_1b_2c_1)\boldsymbol{j}-(a_1b_2c_2)\boldsymbol{i}$$

$$(\boldsymbol{a}.\,\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{b}.\,\boldsymbol{c})\,\boldsymbol{a}=a_1c_1\,(b_1\boldsymbol{i}+b_2\boldsymbol{j})-(b_1c_1+b_2c_2)a_1\boldsymbol{i}$$

$$=(a_1c_1b_2)\boldsymbol{j}-(a_1b_2c_2)\boldsymbol{i}$$

$$\therefore\ (\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=(\boldsymbol{a}.\,\boldsymbol{c})\,\boldsymbol{b}-(\boldsymbol{b}.\,\boldsymbol{c})\boldsymbol{a}$$

(ii)
$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=-\left((\boldsymbol{b}\times\boldsymbol{c})\times\boldsymbol{a}\right)$$

$$=-\left[(\boldsymbol{b}.\,\boldsymbol{a})\boldsymbol{c}-(\boldsymbol{c}.\,\boldsymbol{a})\,\boldsymbol{b}\right]$$

$$=(\boldsymbol{a}.\,\boldsymbol{c})\,\boldsymbol{b}-(\boldsymbol{a}.\,\boldsymbol{b})\,\boldsymbol{c}.$$

5.12.2 Note: *In general, the vector product of three vectors is not associative.*

5.12.3 Corollary: *If* ***a, b*** *are non-collinear vectors and* ***b*** *is perpendicular to neither* ***a*** *nor to* ***c****, then* $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$ *if and only if the vectors* ***a*** *and* ***c*** *are collinear.*

Proof : Suppose $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$

$$\therefore (\boldsymbol{a}.\boldsymbol{c})\,\boldsymbol{b}-(\boldsymbol{b}.\,\boldsymbol{c})\,\boldsymbol{a}=(\boldsymbol{a}.\boldsymbol{c})\,\boldsymbol{b}-(\boldsymbol{a}.\,\boldsymbol{b})\,\boldsymbol{c}$$

$$\therefore (\boldsymbol{b}.\boldsymbol{c})\boldsymbol{a}=(\boldsymbol{a}.\,\boldsymbol{b})\boldsymbol{c}$$

$\therefore$ ***a***, ***c*** are collinear vectors.

Conversely suppose ***a***, ***c*** are collinear vectors, and $\boldsymbol{c}=\lambda\,\boldsymbol{a}$.

$$\therefore (\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}-\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\times\boldsymbol{b})\times(\lambda\boldsymbol{a})-\boldsymbol{a}\times(\boldsymbol{b}\times\lambda\boldsymbol{a})$$

$$=\lambda\left[(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{a}-(\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{a}))\right]$$

$$=\lambda\left[(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{a}-(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{a}\right]\ (\because \boldsymbol{b}\times\boldsymbol{a}=-(\boldsymbol{a}\times\boldsymbol{b}))$$

$$=\lambda\,(\boldsymbol{0})=\boldsymbol{0}\,.$$

5.12.4 Theorem : *If* ***b*** *is perpendicular to both* ***a*** *and* ***c***, *then*

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}\ =\ \boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}).$$

Proof : Suppose ***b*** is perpendicular to ***a*** and ***c***.

Then $\boldsymbol{a}.\boldsymbol{b}=0=\boldsymbol{b}.\boldsymbol{c}$

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=(\boldsymbol{a}.\boldsymbol{c})\,\boldsymbol{b}-(\boldsymbol{b}.\,\boldsymbol{c})\,\boldsymbol{a}=(\boldsymbol{a}.\,\boldsymbol{c})\,\boldsymbol{b}$$

and $$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}.\,\boldsymbol{c})\,\boldsymbol{b}-(\boldsymbol{a}.\,\boldsymbol{b})\,\boldsymbol{c}=(\boldsymbol{a}.\,\boldsymbol{c})\,\boldsymbol{b}$$

Thus $$\boxed{(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})}$$

5.12.5 Theorem : *For any four vectors* ***a***, ***b***, ***c*** *and* ***d*** $(\boldsymbol{a}\times\boldsymbol{b}).(\boldsymbol{c}\times\boldsymbol{d})=\begin{vmatrix}\boldsymbol{a}.\boldsymbol{c} & \boldsymbol{a}.\,\boldsymbol{d}\\ \boldsymbol{b}.\boldsymbol{c} & \boldsymbol{b}.\boldsymbol{d}\end{vmatrix}$ *and in particular*

$$(\boldsymbol{a}\times\boldsymbol{b})^2=\boldsymbol{a}^2\,\boldsymbol{b}^2-(\boldsymbol{a}\,.\,\boldsymbol{b})^2$$

Proof: $$(\boldsymbol{a}\times\boldsymbol{b}).(\boldsymbol{c}\times\boldsymbol{d})=\boldsymbol{a}.(\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{d}))\quad\text{(by 5.10.5)}$$

$$=\boldsymbol{a}.\left[(\boldsymbol{b}\,.\,\boldsymbol{d})\boldsymbol{c}-(\boldsymbol{b}.\boldsymbol{c})\boldsymbol{d}\right]\quad\text{(by 5.12.1)}$$

$$=(\boldsymbol{a}\,.\,\boldsymbol{c})\,(\boldsymbol{b}\,.\,\boldsymbol{d})-(\boldsymbol{a}\,.\,\boldsymbol{d})(\boldsymbol{b}\,.\,\boldsymbol{c})$$

$$=\begin{vmatrix}\boldsymbol{a}.\boldsymbol{c} & \boldsymbol{a}.\,\boldsymbol{d}\\ \boldsymbol{b}.\boldsymbol{c} & \boldsymbol{b}.\boldsymbol{d}\end{vmatrix}$$

In the above formula if $\boldsymbol{c}=\boldsymbol{a}$, and $\boldsymbol{d}=\boldsymbol{b}$, then

$$(\boldsymbol{a}\times\boldsymbol{b}).(\boldsymbol{a}\times\boldsymbol{b})=\begin{vmatrix}\boldsymbol{a}.\boldsymbol{a} & \boldsymbol{a}.\,\boldsymbol{b}\\ \boldsymbol{b}.\boldsymbol{a} & \boldsymbol{b}.\boldsymbol{b}\end{vmatrix}$$

$$=(\boldsymbol{a}.\boldsymbol{a})(\boldsymbol{b}.\boldsymbol{b})-(\boldsymbol{a}.\boldsymbol{b})^2$$

$$=\boldsymbol{a}^2\,\boldsymbol{b}^2-(\boldsymbol{a}\,.\,\boldsymbol{b})^2\,.$$

5.13 Solved Problems

1. Problem: *Prove that the vectors* $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - 3\boldsymbol{j} - 5\boldsymbol{k}$ *and* $\boldsymbol{c} = 3\boldsymbol{i} - 4\boldsymbol{j} - 4\boldsymbol{k}$ *are coplanar.*

Solution: $(\boldsymbol{a} \times \boldsymbol{b}).\boldsymbol{c} = \begin{vmatrix} 2 & -1 & 1 \\ 1 & -3 & -5 \\ 3 & -4 & -4 \end{vmatrix}$ (by Theorem 5.10.8)

$= 2(12 - 20) + (-4 + 15) + (-4 + 9)$

$= -16 + 11 + 5 = 0$

$\therefore$ **a, b, c** are coplanar vectors (Cor. 5.10.9).

2. Problem: *Find the volume of the parallelopiped whose coterminus edges are represented by the vectors* $2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ *and* $2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$.

Solution: Let $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$

$$(\boldsymbol{a} \times \boldsymbol{b}).\boldsymbol{c} = \begin{vmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix}$$

$= 2(1 - 2) + 3(-1 - 4) + 1(1 + 2)$

$= -2 - 15 + 3 = -14$

$\therefore$ volume $= |(\boldsymbol{a} \times \boldsymbol{b}).\boldsymbol{c}| = 14.$

3. Problem: *If the vectors* $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$ *and* $\boldsymbol{c} = 3\boldsymbol{i} + p\boldsymbol{j} + 5\boldsymbol{k}$ *are coplanar, then find* p.

Solution: It is known that **a, b, c** are coplanar if and only if $[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}] = 0$.

(Theorem 5.10.10)

$$\therefore 0 = [\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}] = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & p & 5 \end{vmatrix} = 2(10 + 3p) + 1(5 + 9) + (p - 6)$$

$= 20 + 6p + 14 + p - 6 = 7p + 28$

$\therefore p = -4$.

4. Problem: *Show that* $\boldsymbol{i} \times (\boldsymbol{a} \times \boldsymbol{i}) + \boldsymbol{j} \times (\boldsymbol{a} \times \boldsymbol{j}) + \boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{a}) = 2\boldsymbol{a}$ *for any vector* **a**.

Solution: $\boldsymbol{i} \times (\boldsymbol{a} \times \boldsymbol{i}) = (\boldsymbol{i}.\boldsymbol{i})\boldsymbol{a} - (\boldsymbol{i}.\boldsymbol{a})\boldsymbol{i} = \boldsymbol{a} - (\boldsymbol{i}.\boldsymbol{a})\boldsymbol{i}$

$\boldsymbol{j} \times (\boldsymbol{a} \times \boldsymbol{j}) = \boldsymbol{a} - (\boldsymbol{j}.\boldsymbol{a})\boldsymbol{j}$

$\boldsymbol{k} \times (\boldsymbol{a} \times \boldsymbol{k}) = \boldsymbol{a} - (\boldsymbol{k}.\boldsymbol{a})\boldsymbol{k}$

$\therefore \boldsymbol{i} \times (\boldsymbol{a} \times \boldsymbol{i}) + \boldsymbol{j} \times (\boldsymbol{a} \times \boldsymbol{j}) + \boldsymbol{k} \times (\boldsymbol{a} \times \boldsymbol{k}) = 3\boldsymbol{a} - [(\boldsymbol{i}.\boldsymbol{a})\boldsymbol{i} + (\boldsymbol{j}.\boldsymbol{a})\boldsymbol{j} + (\boldsymbol{k}.\boldsymbol{a})\boldsymbol{k}]$

$= 3\boldsymbol{a} - \boldsymbol{a} = 2\boldsymbol{a}$

$(\because \boldsymbol{a} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \Rightarrow x = \boldsymbol{i}.\boldsymbol{a},\ y = \boldsymbol{a}.\boldsymbol{j},\ z = \boldsymbol{a}.\boldsymbol{k})$.

5. Problem : *Prove that for any three vectors* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, $[\boldsymbol{b}+\boldsymbol{c} \;\; \boldsymbol{c}+\boldsymbol{a} \;\; \boldsymbol{a}+\boldsymbol{b}] = 2[\boldsymbol{a}\;\boldsymbol{b}\;\boldsymbol{c}]$.

Solution: $[\boldsymbol{b}+\boldsymbol{c} \;\; \boldsymbol{c}+\boldsymbol{a} \;\; \boldsymbol{a}+\boldsymbol{b}]$

$$= (\boldsymbol{b}+\boldsymbol{c}).\{(\boldsymbol{c}+\boldsymbol{a})\times(\boldsymbol{a}+\boldsymbol{b})\}$$

$$= (\boldsymbol{b}+\boldsymbol{c}).\{\boldsymbol{c}\times\boldsymbol{a} + \boldsymbol{c}\times\boldsymbol{b} + \boldsymbol{a}\times\boldsymbol{b}\}$$

$$= \boldsymbol{b}.(\boldsymbol{c}\times\boldsymbol{a}) + \boldsymbol{b}.(\boldsymbol{c}\times\boldsymbol{b}) + \boldsymbol{b}.(\boldsymbol{a}\times\boldsymbol{b})$$

$$+ \boldsymbol{c}.(\boldsymbol{c}\times\boldsymbol{a}) + \boldsymbol{c}.(\boldsymbol{c}\times\boldsymbol{b}) + \boldsymbol{c}.(\boldsymbol{a}\times\boldsymbol{b})$$

$$= [\boldsymbol{b}\;\boldsymbol{c}\;\boldsymbol{a}] + 0 + 0 + 0 + 0 + [\boldsymbol{c}\;\boldsymbol{a}\;\boldsymbol{b}]$$

$$= 2[\boldsymbol{a}\;\boldsymbol{b}\;\boldsymbol{c}].$$

6. Problem: *For any three vectors* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, *prove that* $[\boldsymbol{b}\times\boldsymbol{c} \;\; \boldsymbol{c}\times\boldsymbol{a} \;\; \boldsymbol{a}\times\boldsymbol{b}] = [\boldsymbol{a}\;\boldsymbol{b}\;\boldsymbol{c}]^2$.

Solution: $[\boldsymbol{b}\times\boldsymbol{c} \;\; \boldsymbol{c}\times\boldsymbol{a} \;\; \boldsymbol{a}\times\boldsymbol{b}] = (\boldsymbol{b}\times\boldsymbol{c}).\{(\boldsymbol{c}\times\boldsymbol{a})\times(\boldsymbol{a}\times\boldsymbol{b})\}$

$$= (\boldsymbol{b}\times\boldsymbol{c})\cdot[\{(\boldsymbol{c}\times\boldsymbol{a})\cdot\boldsymbol{b}\}\boldsymbol{a} - \{(\boldsymbol{c}\times\boldsymbol{a})\cdot\boldsymbol{a}\}\boldsymbol{b}]$$

$$= (\boldsymbol{b}\times\boldsymbol{c}).\{[\boldsymbol{c}\;\boldsymbol{a}\;\boldsymbol{b}]\boldsymbol{a} - [\boldsymbol{c}\;\boldsymbol{a}\;\boldsymbol{a}]\boldsymbol{b}\}$$

$$= (\boldsymbol{b}\times\boldsymbol{c}).\boldsymbol{a}\,[\boldsymbol{c}\;\boldsymbol{a}\;\boldsymbol{b}] = [\boldsymbol{a}\;\boldsymbol{b}\;\boldsymbol{c}]^2.$$

7. Problem: *Let* $\boldsymbol{a}, \boldsymbol{b}$ *and* $\boldsymbol{c}$ *be unit vectors such that* $\boldsymbol{b}$ *is not parallel to* $\boldsymbol{c}$ *and* $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \frac{1}{2}\boldsymbol{b}$. *Find the angles made by* $\boldsymbol{a}$ *with each of* $\boldsymbol{b}$ *and* $\boldsymbol{c}$.

Solution : $\frac{1}{2}\boldsymbol{b} = \boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}.\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}.\boldsymbol{b})\boldsymbol{c}$

Since $\boldsymbol{b}$ and $\boldsymbol{c}$ are non collinear vectors, equating corresponding coefficients on both sides, $\boldsymbol{a}.\boldsymbol{c} = \frac{1}{2}$ and $\boldsymbol{a}.\boldsymbol{b} = 0$.

$\therefore \boldsymbol{a}$ makes angle $\frac{\pi}{3}$ with $\boldsymbol{c}$ and is perpendicular to $\boldsymbol{b}$.

8. Problem: *Let* $\boldsymbol{a} = \boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$, $\boldsymbol{c} = \boldsymbol{i}-\boldsymbol{j}$ *and* $\boldsymbol{d} = 6\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}$. *Express* $\boldsymbol{d}$, *interms of* $\boldsymbol{b}\times\boldsymbol{c}$, $\boldsymbol{c}\times\boldsymbol{a}$ *and* $\boldsymbol{a}\times\boldsymbol{b}$.

Solution: $[\boldsymbol{a}\;\boldsymbol{b}\;\boldsymbol{c}] = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & -1 & 0 \end{vmatrix}$

$$= 1(0+3) - 1(0-3) + 1(-2+1) = 5.$$

$\boldsymbol{d}.\boldsymbol{a} = 11$, $\boldsymbol{d}.\boldsymbol{b} = 19$, $\boldsymbol{d}.\boldsymbol{c} = 4$

If $\boldsymbol{d} = x(\boldsymbol{b}\times\boldsymbol{c}) + y(\boldsymbol{c}\times\boldsymbol{a}) + z(\boldsymbol{a}\times\boldsymbol{b})$, then we have

$$x = \frac{\boldsymbol{d}\,.\boldsymbol{a}}{[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]},\ y = \frac{\boldsymbol{d}\,.\boldsymbol{b}}{[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]},\ z = \frac{\boldsymbol{d}\,.\boldsymbol{c}}{[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]}\ .$$

$$\therefore\ x = \frac{11}{5},\ y = \frac{19}{5},\ z = \frac{4}{5}$$

$$\therefore\ \boldsymbol{d} = \frac{11}{5}(3\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}) + \frac{19}{5}(-\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) + \frac{4}{5}(4\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}).$$

9. Problem: *For any four vectors* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ *and* $\boldsymbol{d}$*, prove that*

$$(\boldsymbol{b}\times\boldsymbol{c}).(\boldsymbol{a}\times\boldsymbol{d}) + (\boldsymbol{c}\times\boldsymbol{a}).(\boldsymbol{b}\times\boldsymbol{d}) + (\boldsymbol{a}\times\boldsymbol{b}).(\boldsymbol{c}\times\boldsymbol{d}) = 0.$$

Solution: We have $(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) = \boldsymbol{a}\cdot(\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{d}))$

$$= \boldsymbol{a}\cdot((\boldsymbol{b}\cdot\boldsymbol{d})\boldsymbol{c} - (\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{d})$$

$$= (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) - (\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c}) = \begin{vmatrix} \boldsymbol{a}\cdot\boldsymbol{c} & \boldsymbol{a}\cdot\boldsymbol{d} \\ \boldsymbol{b}\cdot\boldsymbol{c} & \boldsymbol{b}\cdot\boldsymbol{d} \end{vmatrix}.$$

Then L.H.S. $= \begin{vmatrix} \boldsymbol{b.a} & \boldsymbol{b.d} \\ \boldsymbol{c.a} & \boldsymbol{c.d} \end{vmatrix} + \begin{vmatrix} \boldsymbol{c.b} & \boldsymbol{c.d} \\ \boldsymbol{a.b} & \boldsymbol{a.d} \end{vmatrix} + \begin{vmatrix} \boldsymbol{a.c} & \boldsymbol{a.d} \\ \boldsymbol{b.c} & \boldsymbol{b.d} \end{vmatrix}$

$$= (\boldsymbol{b}\,.\,\boldsymbol{a})(\boldsymbol{c}\,.\,\boldsymbol{d}) - (\boldsymbol{b}\,.\,\boldsymbol{d})(\boldsymbol{c}\,.\,\boldsymbol{a}) + (\boldsymbol{c}\,.\,\boldsymbol{b})(\boldsymbol{a}\,.\,\boldsymbol{d}) - (\boldsymbol{a}\,.\,\boldsymbol{b})(\boldsymbol{c}\,.\,\boldsymbol{d})$$

$$+ (\boldsymbol{a}\,.\,\boldsymbol{c})(\boldsymbol{b}\,.\,\boldsymbol{d}) - (\boldsymbol{a}\,.\,\boldsymbol{d})(\boldsymbol{b}\,.\,\boldsymbol{c}) = 0.$$

10. Problem: *Find the equation of the plane passing through the points* A = (2, 3, –1), B = (4, 5, 2) *and* C = (3, 6, 5).

Solution : Let 'O' be the origin. Let $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ be the position vector of any point P in the plane of ΔABC. Then the vectors **AP, AB, AC** are coplanar.

$$\therefore\ [\mathbf{AP}\ \ \mathbf{AB}\ \ \mathbf{AC}] = 0.$$

Now $\mathbf{AP} = (x-2,\ y-3,\ z+1)$

$\mathbf{AB} = (2, 2, 3)$ and $\mathbf{AC} = (1, 3, 6)$

$$\therefore\ [\mathbf{AP}\ \mathbf{AB}\ \mathbf{AC}] = 0 \Rightarrow \begin{vmatrix} x-2 & y-3 & z+1 \\ 2 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} = 0.$$

i.e., $(x-2)(12-9) - (y-3)(12-3) + (z+1)(6-2) = 0$

i.e., $3x - 9y + 4z + 25 = 0.$

11. Problem: *Find the equation of the plane passing through the point* A = (3, –2, –1) *and parallel to the vectors* $\boldsymbol{b} = \boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}$ *and* $\boldsymbol{c} = 3\boldsymbol{i} + 2\boldsymbol{j} - 5\boldsymbol{k}$.

Solution: Let $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ be the position vector of any point P in the given plane.

Then $[\boldsymbol{r}-\boldsymbol{a}\ \ \boldsymbol{b}\ \ \boldsymbol{c}] = 0$ (Theorem 5.11.1)

$$\therefore\ \begin{vmatrix} x-3 & y+2 & z+1 \\ 1 & -2 & 4 \\ 3 & 2 & -5 \end{vmatrix} = 0.$$

$\therefore \quad (x-3)(10-8) - (y+2)(-5-12) + (z+1)(2+6) = 0$

$\Rightarrow \; 2x + 17y + 8z + 36 = 0.$

12. Problem: *Find the vector equation of the plane passing through the intersection of the planes* $\boldsymbol{r}\,.\,(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}) = 6$ *and* $\boldsymbol{r}\,.\,(2\boldsymbol{i}+3\boldsymbol{j}+4\boldsymbol{k}) = -5$ *and the point* (1, 1, 1).

Solution: Here $\boldsymbol{n}_1 = \boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{n}_2 = 2\boldsymbol{i}+3\boldsymbol{j}+4\boldsymbol{k}$; Also $d_1 = 6$ and $d_2 = -5$.

Substituting these values in the relation $\boldsymbol{r}.(\boldsymbol{n}_1+\lambda\boldsymbol{n}_2) = d_1+\lambda d_2$, we get

$$\boldsymbol{r}.[(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}+\lambda(2\boldsymbol{i}+3\boldsymbol{j}+4\boldsymbol{k})] = 6-5\lambda$$

or $\boldsymbol{r}\,.\,[(1+2\lambda)\,\boldsymbol{i}+(1+3\lambda)\,\boldsymbol{j}+(1+4\lambda)\boldsymbol{k}] = 6-5\lambda,$ ...(1)

where λ is some real number.

Taking $\boldsymbol{r} = x\,\boldsymbol{i}+y\,\boldsymbol{j}+z\boldsymbol{k}$, we get

$(x\,\boldsymbol{i}+y\,\boldsymbol{j}+z\,\boldsymbol{k})\,.\,[(1+2\lambda)\boldsymbol{i}+(1+3\lambda)\,\boldsymbol{j}+(1+4\lambda)\boldsymbol{k}] = 6-5\lambda$

or $(1+2\lambda)x+(1+3\lambda)\,y+(1+4\lambda)z = 6-5\lambda$

or $(x+y+z-6)+\lambda(2x+3y+4z+5) = 0$ ...(2)

Since this plane passes through the point (1, 1, 1), it should satisfy this equation (2). Then

$(1+1+1-6)+\lambda(2+3+4+5) = 0 \Rightarrow \lambda = \dfrac{3}{14}.$

Substituting this value of λ in equation (1), we get

$$\boldsymbol{r}\,.\left[\left(1+\frac{3}{7}\right)\boldsymbol{i}+\left(1+\frac{9}{14}\right)\boldsymbol{j}+\left(1+\frac{6}{7}\right)\boldsymbol{k}\right] = 6-\frac{15}{14}$$

or $\boldsymbol{r}\,.\left(\dfrac{10}{7}\boldsymbol{i}+\dfrac{23}{14}\boldsymbol{j}+\dfrac{13}{7}\boldsymbol{k}\right) = \dfrac{69}{14}$ or $\boldsymbol{r}\,.\,(20\boldsymbol{i}+23\boldsymbol{j}+26\boldsymbol{k}) = 69$, which is the required vector equation of the plane.

13. Problem: *Find the distance of a point (2, 5, –3) from the plane* $\boldsymbol{r}.(6\boldsymbol{i}-3\boldsymbol{j}+2\boldsymbol{k}) = 4.$

Solution: Here $\boldsymbol{a} = 2\boldsymbol{i}+5\boldsymbol{j}-3\boldsymbol{k}$, $\mathbf{N} = 6\,\boldsymbol{i}-3\boldsymbol{j}+2\boldsymbol{k}$; and $d = 4$.

$\therefore$ The distance of the point (2, 5, –3) from the given plane is

$$\frac{|(2\boldsymbol{i}+5\boldsymbol{j}-3\boldsymbol{k})\,.(6\boldsymbol{i}-3\boldsymbol{j}+2\boldsymbol{k})-4|}{|6\boldsymbol{i}-3\boldsymbol{j}+2\boldsymbol{k}|} = \frac{|12-15-6-4|}{\sqrt{36+9+4}} = \frac{13}{7}.$$

14. Problem: *Find the angle between the line* $\dfrac{x+1}{2} = \dfrac{y}{3} = \dfrac{z-3}{6}$ *and the plane* $10x+2y-11z = 3$.

Solution: Let ϕ be the angle between the given line and the normal to the plane.

Converting the given equations into vector form, we have

$$\boldsymbol{r} = (-\boldsymbol{i}+3\,\boldsymbol{k})+\lambda\,(2\boldsymbol{i}+3\boldsymbol{j}+6\boldsymbol{k})$$

and $\boldsymbol{r}\,.\,(10\boldsymbol{i}+2\boldsymbol{j}-11\boldsymbol{k}) = 3.$

Here $\boldsymbol{b} = 2\boldsymbol{i}+3\boldsymbol{j}+6\boldsymbol{k}$ and $\boldsymbol{n} = 10\boldsymbol{i}+2\boldsymbol{j}-11\boldsymbol{k}.$

$$\sin\phi = \left|\frac{(2\boldsymbol{i}+3\boldsymbol{j}+6\boldsymbol{k}).(10\boldsymbol{i}+2\boldsymbol{j}-11\boldsymbol{k})}{\sqrt{2^2+3^2+6^2}\sqrt{10^2+2^2+11^2}}\right|$$

$$= \left|\frac{-40}{7\times 15}\right| = \frac{8}{21} \text{ or } \phi = \sin^{-1}\left(\frac{8}{21}\right).$$

15. Problem: *For any four vectors* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ *and* $\boldsymbol{d}$, $(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d}) = [\boldsymbol{a}\ \boldsymbol{c}\ \boldsymbol{d}]\boldsymbol{b} - [\boldsymbol{b}\ \boldsymbol{c}\ \boldsymbol{d}]\boldsymbol{a}$ *and* $(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d}) = [\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{d}]\boldsymbol{c} - [\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]\boldsymbol{d}$.

Solution: Let $\boldsymbol{m} = \boldsymbol{c}\times\boldsymbol{d}$

$$\therefore (\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d}) = (\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{m}$$

$$= (\boldsymbol{a}.\boldsymbol{m})\boldsymbol{b} - (\boldsymbol{b}.\boldsymbol{m})\boldsymbol{a}$$

$$= (\boldsymbol{a}.(\boldsymbol{c}\times\boldsymbol{d}))\boldsymbol{b} - (\boldsymbol{b}.(\boldsymbol{c}\times\boldsymbol{d}))\boldsymbol{a}$$

$$= [\boldsymbol{a}\ \boldsymbol{c}\ \boldsymbol{d}]\boldsymbol{b} - [\boldsymbol{b}\ \boldsymbol{c}\ \boldsymbol{d}]\boldsymbol{a}.$$

Again Let $\boldsymbol{a}\times\boldsymbol{b} = \boldsymbol{n}$.

Then $(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d}) = \boldsymbol{n}\times(\boldsymbol{c}\times\boldsymbol{d})$

$$= (\boldsymbol{n}.\boldsymbol{d})\boldsymbol{c} - (\boldsymbol{n}.\boldsymbol{c})\boldsymbol{d}$$

$$= ((\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{d})\boldsymbol{c} - ((\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c})\boldsymbol{d}$$

$$= [\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{d}]\boldsymbol{c} - [\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]\boldsymbol{d}.$$

16. Problem: *Find the shortest distance between the skew lines* $\boldsymbol{r} = (6\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k}) + t(\boldsymbol{i}-2\boldsymbol{j}+2\boldsymbol{k})$ *and* $\boldsymbol{r} = (-4\boldsymbol{i}-\boldsymbol{k}) + s(3\boldsymbol{i}-2\boldsymbol{j}-2\boldsymbol{k})$.

Solution: The first line passes through the point A(6, 2, 2) and is parallel to the vector $\boldsymbol{b} = \boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$. Second line passes through the point C(–4, 0, –1) and is parallel to the vector $\boldsymbol{d} = 3\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$. (Fig. 5.29).

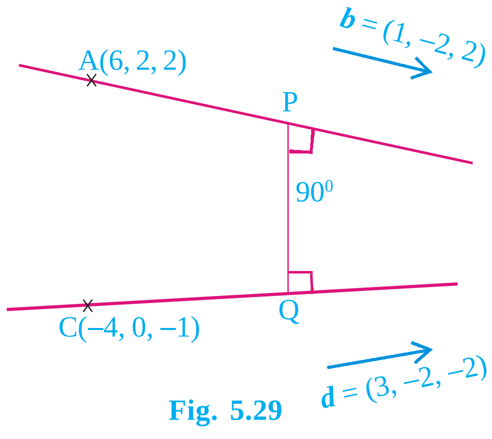


Fig. 5.29

Shortest distance $= \dfrac{|[AC\ \boldsymbol{b}\ \boldsymbol{d}]|}{|\boldsymbol{b}\times\boldsymbol{d}|}$ (Theorem 5.11.7)

$$[AC\ \boldsymbol{b}\ \boldsymbol{d}] = \begin{vmatrix} -10 & -2 & -3 \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = -108;$$

$$\boldsymbol{b}\times\boldsymbol{d} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = 8\boldsymbol{i} + 8\boldsymbol{j} + 4\boldsymbol{k} \text{ and } |\boldsymbol{b}\times\boldsymbol{d}| = 12$$

$$\therefore \text{Shortest distance between the skew lines} = \frac{|[AC\ \boldsymbol{b}\ \boldsymbol{d}]|}{|\boldsymbol{b}\times\boldsymbol{d}|} = \frac{108}{12} = 9.$$

Exercise 5(c)

I. 1. Compute $[\boldsymbol{i}-\boldsymbol{j} \quad \boldsymbol{j}-\boldsymbol{k} \quad \boldsymbol{k}-\boldsymbol{i}]$.

2. If $\boldsymbol{a}=\boldsymbol{i}-2\boldsymbol{j}-3\boldsymbol{k}$, $\boldsymbol{b}=2\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}$, $\boldsymbol{c}=\boldsymbol{i}+3\boldsymbol{j}-2\boldsymbol{k}$, then compute $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$.

3. If $\boldsymbol{a}=(1,-1,-6)$, $\boldsymbol{b}=(1,-3,4)$ and $\boldsymbol{c}=(2,-5,3)$, then compute the following. (i) $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ (ii) $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$ (iii) $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}$.

4. Simplify the following

(i) $(\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k})\times(2\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}).(\boldsymbol{j}+\boldsymbol{k})$

(ii) $(2\boldsymbol{i}-3\boldsymbol{j}+\boldsymbol{k}).(\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k})\times(2\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})$

5. Find the volume of the parallelopiped having coterminus edges $\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$, $\boldsymbol{i}-\boldsymbol{j}$ and $\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$.

6. Find t for which the vectors $2\boldsymbol{i}-3\boldsymbol{j}+\boldsymbol{k}$, $\boldsymbol{i}+2\boldsymbol{j}-3\boldsymbol{k}$ and $\boldsymbol{j}-t\boldsymbol{k}$ are coplanar.

7. For non-coplanar vectors, $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, determine p for which the vectors $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$, $\boldsymbol{a}+p\boldsymbol{b}+2\boldsymbol{c}$ and $-\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$ are coplanar.

8. Determine λ, for which the volume of the parallelopiped having coterminus edges $\boldsymbol{i}+\boldsymbol{j}$, $3\boldsymbol{i}-\boldsymbol{j}$ and $3\boldsymbol{j}+\lambda\boldsymbol{k}$ is 16 cubic units.

9. Find the volume of the tetrahedron having the edges $\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$, $\boldsymbol{i}-\boldsymbol{j}$ and $\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$.

10. Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be non-coplanar vectors and $\boldsymbol{\alpha}=\boldsymbol{a}+2\boldsymbol{b}+3\boldsymbol{c}$, $\boldsymbol{\beta}=2\boldsymbol{a}+\boldsymbol{b}-2\boldsymbol{c}$ and $\boldsymbol{\gamma}=3\boldsymbol{a}-7\boldsymbol{c}$, then find $[\boldsymbol{\alpha}\,\boldsymbol{\beta}\,\boldsymbol{\gamma}]$.

11. Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be non-coplanar vectors. If $[2\boldsymbol{a}-\boldsymbol{b}+3\boldsymbol{c},\ \boldsymbol{a}+\boldsymbol{b}-2\boldsymbol{c},\ \boldsymbol{a}+\boldsymbol{b}-3\boldsymbol{c}]=\lambda[\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}]$, then find the value of λ.

12. Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be non-coplanar vectors. If $[\boldsymbol{a}+2\boldsymbol{b}\ \ 2\boldsymbol{b}+\boldsymbol{c}\ \ 5\boldsymbol{c}+\boldsymbol{a}]=\lambda[\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}]$, then find λ.

13. If $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are non-coplanar vectors, then find the value of

$$\frac{(\boldsymbol{a}+2\boldsymbol{b}-\boldsymbol{c}).\left[(\boldsymbol{a}-\boldsymbol{b})\times(\boldsymbol{a}-\boldsymbol{b}-\boldsymbol{c})\right]}{[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]}.$$

14. If $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are mutually perpendicular unit vectors, then find the value of $[\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}]^2$.

15. $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are non-zero vectors and $\boldsymbol{a}$ is perpendicular to both $\boldsymbol{b}$ and $\boldsymbol{c}$. If $|\boldsymbol{a}|=2$, $|\boldsymbol{b}|=3$, $|\boldsymbol{c}|=4$ and $(\boldsymbol{b},\boldsymbol{c})=\dfrac{2\pi}{3}$, then find $|[\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}]|$.

16. If $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are unit coplanar vectors, then find $[2\boldsymbol{a}-\boldsymbol{b},\ 2\boldsymbol{b}-\boldsymbol{c},\ 2\boldsymbol{c}-\boldsymbol{a}]$.

II. 1. If $[\boldsymbol{b}\ \boldsymbol{c}\ \boldsymbol{d}]+[\boldsymbol{c}\ \boldsymbol{a}\ \boldsymbol{d}]+[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{d}]=[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]$, then show that the points with position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ are coplanar.

2. If $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are non-coplanar vectors, then prove that the four points with position vectors $2\boldsymbol{a}+3\boldsymbol{b}-\boldsymbol{c}$, $\boldsymbol{a}-2\boldsymbol{b}+3\boldsymbol{c}$, $3\boldsymbol{a}+4\boldsymbol{b}-2\boldsymbol{c}$ and $\boldsymbol{a}-6\boldsymbol{b}+6\boldsymbol{c}$ are coplanar.

3. $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are non-zero and non-collinear vectors and $\theta \neq 0$, is the angle between $\boldsymbol{b}$ and $\boldsymbol{c}$. If $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \frac{1}{3}|\boldsymbol{b}|\,|\boldsymbol{c}|\,\boldsymbol{a}$, then find $\sin\theta$.

4. Find the volume of the tetrahedron whose vertices are (1, 2, 1), (3, 2, 5), (2, –1, 0) and (–1, 0, 1).

5. Show that $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{b} + \boldsymbol{c}) \times (\boldsymbol{c} + \boldsymbol{a}) = 2[\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}]$.

6. Show that the equation of the plane passing through the points with position vectors $3i - 5j - k$, $-i + 5j + 7k$ and parallell to the vector $3i - j + 7k$ is $3x + 2y - z = 0$.

7. Prove that $\boldsymbol{a} \times [\boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{b})] = (\boldsymbol{a} \cdot \boldsymbol{a})(\boldsymbol{b} \times \boldsymbol{a})$.

8. If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{d}$ are coplanar vectors, then show that $(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = \boldsymbol{0}$.

9. Show that $[(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{a} \times \boldsymbol{c})] \cdot \boldsymbol{d} = (\boldsymbol{a} \cdot \boldsymbol{d})[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]$.

10. Show that $\boldsymbol{a} \cdot [(\boldsymbol{b} + \boldsymbol{c}) \times (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})] = 0$.

11. Find λ in order that the four points A(3, 2, 1), B (4, λ, 5), C (4, 2, –2) and D(6, 5, –1) be coplanar.

12. Find the vector equation of the plane passing through the intersection of planes $\boldsymbol{r} \cdot (2\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}) = 7$, $\boldsymbol{r} \cdot (2\boldsymbol{i} + 5\boldsymbol{j} + 3\boldsymbol{k}) = 9$ and through the point (2, 1, 3).

13. Find the equation of the plane passing through (a, b, c) and parallel to the plane $\boldsymbol{r} \cdot (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = 2$.

14. Find the shortest distance between the lines $\boldsymbol{r} = 6\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k} + \lambda(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$ and $\boldsymbol{r} = -4\boldsymbol{i} - \boldsymbol{k} + \mu(3\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k})$.

15. Find the equation of the plane passing through the line of intersection of the planes $\boldsymbol{r} \cdot (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = 1$ and $\boldsymbol{r} \cdot (2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}) + 4 = 0$ and parallel to X-axis.

16. Prove that the four points $4\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k}$, $-(\boldsymbol{j} + \boldsymbol{k})$, $3\boldsymbol{i} + 9\boldsymbol{j} + 4\boldsymbol{k}$ and $-4\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}$ are coplanar.

17. If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are non-coplanar, then show that the vectors $\boldsymbol{a} - \boldsymbol{b}, \boldsymbol{b} + \boldsymbol{c}, \boldsymbol{c} + \boldsymbol{a}$ are coplanar.

18. If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are the position vectors of the points A, B and C respectively, then prove that the vector $\boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{c} \times \boldsymbol{a}$ is perpendicular to the plane of ΔABC.

III. 1. Show that $(\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})) \times \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \times \boldsymbol{c})$ and $(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{a} \times \boldsymbol{c}) + (\boldsymbol{a} \cdot \boldsymbol{b})(\boldsymbol{a} \cdot \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{c})$.

2. If A = (1, –2, –1), B = (4, 0, –3), C = (1, 2, –1) and D = (2, –4, –5), find the distance between AB and CD.

3. If $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{c} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, find $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ and $|(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}|$.

4. If $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$, verify that $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \neq (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$.

5. If $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{c} = -\boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{k}$ and $\boldsymbol{d} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, then compute $\left|(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d})\right|$.

6. If $\mathbf{A} = (1, a, a^2)$, $\mathbf{B} = (1, b, b^2)$ and $\mathbf{C} = (1, c, c^2)$ are non–coplanar vectors and
$$\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0,$$ then show that $a\,b\,c + 1 = 0$.

7. If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are non-zero vectors, then $|(\boldsymbol{a} \times \boldsymbol{b} \,.\, \boldsymbol{c})| = |\boldsymbol{a}|\,|\boldsymbol{b}|\,|\boldsymbol{c}| \Leftrightarrow \boldsymbol{a}.\boldsymbol{b} = \boldsymbol{b}.\boldsymbol{c} = \boldsymbol{c}.\boldsymbol{a} = 0$.

8. If $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{c} = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ then find $|(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}|$ and $|\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})|$.

9. If $|\boldsymbol{a}| = 1$, $|\boldsymbol{b}| = 1$, $|\boldsymbol{c}| = 2$ and $\boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{c}) + \boldsymbol{b} = \boldsymbol{0}$ then find the angle between $\boldsymbol{a}$ and $\boldsymbol{c}$.

10. Let $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{k}$, $\boldsymbol{b} = x\boldsymbol{i} + \boldsymbol{j} + (1 - x)\boldsymbol{k}$ and $\boldsymbol{c} = y\boldsymbol{i} + x\boldsymbol{j} + (1 + x - y)\boldsymbol{k}$. Prove that the scalar triple product $[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]$ is independent of both x and y.

11. Let $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{c} = \boldsymbol{i} + 3\boldsymbol{k}$. If $\boldsymbol{a}$ is a unit vector then find the maximum value of $[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]$.

12. Let $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j}$, $\boldsymbol{b} = \boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{c} = \boldsymbol{k} - \boldsymbol{i}$. Find unit vector $\boldsymbol{d}$ such that $\boldsymbol{a}.\boldsymbol{d} = 0 = [\boldsymbol{b}\ \boldsymbol{c}\ \boldsymbol{d}]$.

Key Concepts

- Concept of scalar product (or dot) of two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ containing angle 'θ' is introduced as $|\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$ which is geometrically equal to product of the magnitude of one of the vectors and the projection of the other on the first vector.

- If θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, then $\cos\theta = \dfrac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|} = \dfrac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{\sum a_1^2}\sqrt{\sum b_1^2}}$ where
$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$, $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ in $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ system and
$\boldsymbol{a}.\boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

- Right handed and left handed system of vectors. Definition of cross product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as $\boldsymbol{a} \times \boldsymbol{b} = (|\boldsymbol{a}|\,|\boldsymbol{b}| \sin\theta)\,\boldsymbol{n}$ where $\boldsymbol{a}$, $\boldsymbol{b}$ are non-zero and non-collinear vectors, θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ and $\boldsymbol{n}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{n})$ is a right handed system.

❖ Cross product is not commutative, infact for any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, it is proved that $\boldsymbol{b} \times \boldsymbol{a} = -(\boldsymbol{a} \times \boldsymbol{b})$.

❖ If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ then $\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.

❖ If θ is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then $\sin\theta = \dfrac{|\boldsymbol{a} \times \boldsymbol{b}|}{|\boldsymbol{a}||\boldsymbol{b}|}$.

❖ While determining the angle between two vectors, considering $\boldsymbol{a} \cdot \boldsymbol{b}$ is always better.

❖ Introduced the concept of scalar triple product of three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ as $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$ and explained the $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$ is equal to $\pm$ V where 'V' is the volume of the parallelopiped with $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ as coterminus edges according as $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is a right handed system or a left handed system and thus V is the numerical value of $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$.

❖ Introduced the notation $[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]$ for $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$ and proved that $[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}] = [\boldsymbol{b}\ \boldsymbol{c}\ \boldsymbol{a}] = [\boldsymbol{c}\ \boldsymbol{a}\ \boldsymbol{b}]$.

❖ If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$, $b = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ and $c = c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k}$ then

$$[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

❖ A necessary and sufficient condition for three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ to be coplanar is that $[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}] = 0$, equivalently $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$.

❖ **Volume of a tetrahedron** with vertices A, B, C and D is $\dfrac{1}{6}|[\mathbf{AB}\ \mathbf{AC}\ \mathbf{AD}]|$.

❖ Two lines are said to be **skew lines** if there is no plane containing both the lines. The shortest distance between two straight lines $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$ and $\boldsymbol{r} = \boldsymbol{c} + s\boldsymbol{d}$ is $\dfrac{|(\boldsymbol{c} - \boldsymbol{a}) \cdot (\boldsymbol{b} \times \boldsymbol{d})|}{|\boldsymbol{b} \times \boldsymbol{d}|}$.

❖ **Vector equation** of the line passing through the point $\boldsymbol{a}$ and parallel to the vector $\boldsymbol{b}$ *is* $(\boldsymbol{r} - \boldsymbol{a}) \times \boldsymbol{b} = \mathbf{0}$.

❖ $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{c})\,\boldsymbol{a}$ and $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{c}$.

❖ $(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = \begin{vmatrix} \boldsymbol{a} \cdot \boldsymbol{c} & \boldsymbol{a} \cdot \boldsymbol{d} \\ \boldsymbol{b} \cdot \boldsymbol{c} & \boldsymbol{b} \cdot \boldsymbol{d} \end{vmatrix}$ and $(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = [\boldsymbol{a}\ \boldsymbol{c}\ \boldsymbol{d}]\,\boldsymbol{b} - [\boldsymbol{b}\ \boldsymbol{c}\ \boldsymbol{d}]\,\boldsymbol{a}$
$= [\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{d}]\,\boldsymbol{c} - [\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]\,\boldsymbol{d}$.

Historical Note

Vector Analysis came into existence during the fourth decade of 19th century. Preceding the advent of Vector Analysis three events merit mention.

(i) Discovery and geometrical representation of complex numbers.

(ii) Leibnitz's search for a geometry of position.

(iii) The idea of parallelogram law of forces and velocities.

Josiah Willard Gibbs (1839 - 1903) work on Vector Analysis was of major importance in pure mathematics. Using the ideas of *Hermann Grassmann* (1809 –1877), *Gibbs* produced a system more easily applied than that of *Hamilton.*

During 19th century, while *Grassmann's* Hypercomplex numbers were hardly noticed, *Hamilton's* quaternion calculus fell flat in the mathematical world. Except for *Tait* and *Gibbs*, the majority of the scientists preferred to work with the old fashioned *Cartesian* methods. Even as recently as 1930's the vector could hardly be said to have come into its domain.

Answers

Exercise 5(a)

I. **1.** 60^0 **2.** $\lambda = 3$

3. $\lambda = 1$ **4.** $\boldsymbol{c} = -3\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}$

5. $\mathrm{Cos}^{-1}\left(\dfrac{2}{3\sqrt{46}}\right)$ **6.** $\lambda = \dfrac{1}{2}$

7. (i) $2(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}),\ 2\sqrt{3}$, (ii) $2(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}),\ \boldsymbol{j} - \boldsymbol{k}$

8. $\boldsymbol{r}.(4\boldsymbol{i} + 7\boldsymbol{j} - 4\boldsymbol{k}) = -6$ **9.** $\mathrm{Cos}^{-1}\left(\dfrac{52}{\sqrt{74 \times 65}}\right)$

II. **1.** $\pm\dfrac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$ **2.** 60^0 **3.** $\sqrt{29}$

4. $\boldsymbol{r}.(3\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}) = 2$, Cartesian form : $3x - 2y - 2z - 2 = 0$ and the distance of this plane from the origin is $\dfrac{2}{\sqrt{17}}$.

III. **2.** $\pm 7\sqrt{33}(\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$

Exercise 5(b)

I. **1.** 9 **2.** $\sqrt{210}$ **3.** $-2(2\boldsymbol{i} + 5\boldsymbol{j} + 11\boldsymbol{k})$

4. $p = 12$ **5.** **0** **6.** $x^2 + y^2$

7. $-6\boldsymbol{i} - \boldsymbol{j} - 6\boldsymbol{k}$ 8. $\pm\frac{1}{\sqrt{6}}(2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$ 9. $\frac{\sqrt{3}}{2}$

10. 3 11. $5\sqrt{3}$ 12. $\frac{41}{2}$

13. $\pm\frac{1}{7}(-3\boldsymbol{i} + 2\boldsymbol{j} - 6\boldsymbol{k})$ 14. $\frac{3\sqrt{3}}{2}$

II. 2. -54 3. $3\boldsymbol{i} - 4\boldsymbol{j} - 5\boldsymbol{k}, 5\sqrt{2}$ 5. $50\sqrt{2}$

6. $\pm(\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$ 7. $\pm\frac{1}{\sqrt{6}}(2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$ 9. $\pm(2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k})$

10. 25 11. $\pm\frac{1}{\sqrt{10}}(3\boldsymbol{i} + \boldsymbol{j})$

III. 2. $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = 2\boldsymbol{i} + 4\boldsymbol{j} - 4\boldsymbol{k}$

3. $\boldsymbol{a} \times \boldsymbol{b} = -16\boldsymbol{i} - 50\boldsymbol{j} + 4\boldsymbol{k}$, $\boldsymbol{a} \times \boldsymbol{c} = -5\boldsymbol{i} - 4\boldsymbol{j} + 9\boldsymbol{k}$,

$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = -21\boldsymbol{i} - 54\boldsymbol{j} + 13\boldsymbol{k}$

4. $\frac{1}{3}(5\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k})$ 5. 1 7. 2

8. (i) $120\,\boldsymbol{i} + 304\,\boldsymbol{j} + 424\,\boldsymbol{k}$ (ii) -80

Exercise 5(c)

I. 1. 0 2. -20

3. (i) 0, (ii) $29\boldsymbol{i} - 67\boldsymbol{j} + 16\boldsymbol{k}$, (iii) $-40\,\boldsymbol{i} + 62\,\boldsymbol{j} + 130\,\boldsymbol{k}$

4. (i) 12, (ii) -12 5. 5 6. $t = 1$

7. $p = 2$ 8. $\lambda = \pm 4$ 9. 1/6 10. 0

11. $\lambda = -3$ 12. $\lambda = 12$ 13. 3 14. 1

15. $12\sqrt{3}$ 16. 0

II. 3. $\frac{2\sqrt{2}}{3}$ 4. 6 11. $\lambda = 5$

12. $\boldsymbol{r} \cdot (38\boldsymbol{i} + 68\boldsymbol{j} + 3\boldsymbol{k}) = 153$

13. $x + y + z = a + b + c$ 14. 9 15. $y - 3z + 6 = 0$

III. 2. 4 / 3 3. $-9\boldsymbol{i} - 6\boldsymbol{j} - 3\boldsymbol{k}, \sqrt{174}$ 5. $5\sqrt{114}$

8. $5\sqrt{14}, \sqrt{54}$ 9. 30^0 or 150^0 11. $\sqrt{59}$

12. $\boldsymbol{d} = \pm\frac{1}{\sqrt{6}}(\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k})$

Trigonometry

Chapter 6

Trigonometric Ratios upto Transformations

"The acharya (master) title (in astronomy) is offered on him who has acquired sufficient knowledge of Trigonometry"

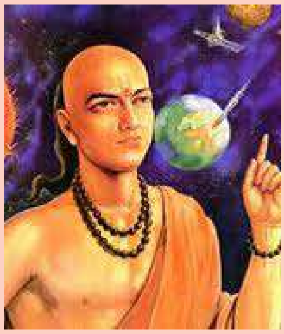
- Bhaskaracharya

Introduction

In the earlier classes we have constructed in geometry, triangles, quadrilaterals, pentagons, hexagons and so on. All these figures are generally called as polygons with n sides. When $n = 5$, it is called a pentagon, when $n = 6$, it is called a hexagon, when $n = 10$ it is called a decagon etc. On the same lines, when $n = 3$, we can call the polygon as 'trigon' (instead of 'triangle'). The word 'trigonometry' can be read as 'trigon-o-metry'. This word is derived from two Greek words

(i) trigonon (ii) metron

The word 'trigonon' means a triangle and the word 'metron' means a measure. Thus trigonometry is the science that deals with measurement of triangles. Trigonometry has great use in measurement of areas, heights, distances etc.



Varahamihira

(505 - 587)

Varahamihira, also called Mihira, was an astronomer-mathematician, born in Ujjain. Varahamihira's picture is found in the Indian Parliament along side Aryabhatta's, of whom he was a follower. He was considered to be one of the nine jewels (Navaratnas) of the court of legendary king Vikramaditya. Varahamihira discovered Pascal's triangle and worked on magic squares. His most famous treatise is 'Pancha Siddhantika' (575 A.D.).

It has many applications in almost all branches of science in general and in Physics and Engineering in particular.

To study properties of triangles, first we should learn properties of the angles of a triangle. Though in geometry the angles of a triangle or a quadrilateral etc. are always less than two right angles, in the study of trigonometry, we do not impose any restriction on the magnitude of an angle. It can be any real number (positive or negative or zero).

According to the great mathematician 'Euclid', angle is 'the inclination of two lines intersecting at a point'. We can formally define angle as follows.

An **'angle'** is the union of two rays having a common end point in a plane. The amount of rotation in the plane that is necessary to bring one ray into the position of the other ray is called the **'magnitude of the angle'**. (see Fig. 6.1). An angle is usually denoted by θ, α etc.

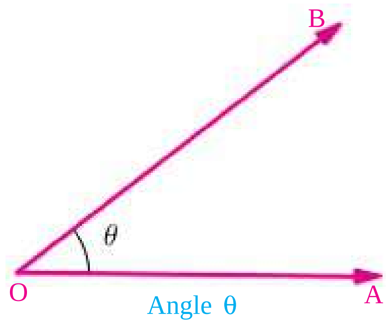


Fig. 6.1

In figure 6.1 angle AOB is θ. $\overrightarrow{OA}$ is called the initial side and $\overrightarrow{OB}$ is called the terminal side of the angle θ. In the process of rotation, $\overrightarrow{OB}$ will be collinear with $\overrightarrow{OA}$ but will have direction opposite to that of $\overrightarrow{OA}$. At this instant the angle formed by the two rays is called a positive straight angle which is shown in Fig. 6.2.

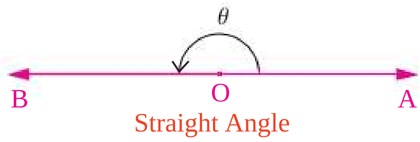


Fig. 6.2

A positive 'right angle' is defined as half of a positive straight angle which is shown in Fig. 6.3.

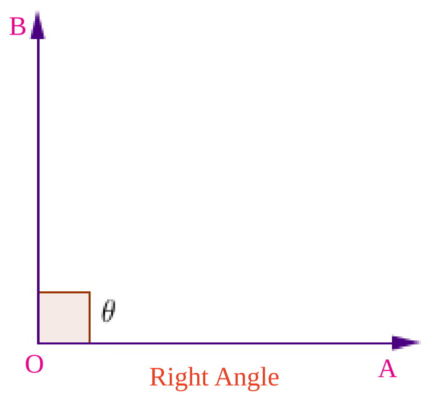


Fig. 6.3

We have learnt, in the previous classes, that there are three systems for the measurement of angles.

1. Sexagesimal system or British system
2. Centisimal system or French system
3. Circular measurement

In the Sexagesimal system

1 right angle	=	90 degrees (90^0)
1 degree	=	60 minutes ($60'$)
1 minute	=	60 seconds ($60''$)

In the Centisimal system

1 right angle	=	100 grades (100^g)
1 grade	=	100 minutes ($100'$)
1 minute	=	100 seconds ($100''$)

In the circular measure, **one radian** is defined as the amount of the angle subtended by an arc of length '*r*' units of a circle of radius '*r*' units at the centre of that circle. This angle is independent of the size of the circle (i.e., the radius of the circle). One radian is denoted by 1^C. In this measure

$$2 \text{ right angles} = \pi^C$$

Though we have used the same name 'minute' (or 'second') in both 'sexagesimal system' and 'centisimal system', it can be easily observed that they are not same.

1 minute in the sexagesimal system = $\frac{1}{90 \times 60}th$ of a right angle where as

1 minute in the centisimal system = $\frac{1}{100 \times 100}th$ of a right angle.

The conversion from one system to the other can be easily done using the equation :

$$\boxed{\frac{180}{D} = \frac{200}{G} = \frac{\pi}{R}}.$$

where D, G, R respectively denote degrees, grades and radians.

For example, to convert 30^0 into grades and radians, put D = 30 in the above equation and find G, R as follows:

$$\frac{180}{30} = \frac{200}{G} = \frac{\pi}{R}. \qquad \text{Hence } G = \frac{100}{3}, \; R = \frac{30\pi}{180} = \frac{\pi}{6}$$

$$\text{Thus } 30^0 = \frac{100^g}{3} = \frac{\pi^C}{6}.$$

6.1 Trigonometric ratios - variation - Graphs and periodicity

A ratio is $\frac{a}{b}$ where a, b are two real numbers and b is non-zero. If we take a right angled triangle with θ as one of its acute angles, using the lengths a, b, c of the three sides of the triangle (see Fig. 6.4) we can form six ratios, namely, $\frac{b}{c}, \frac{a}{c}, \frac{b}{a}, \frac{c}{b}, \frac{c}{a}, \frac{a}{b}$.

These six ratios are called the trigonometric ratios of the angle θ. Each of these ratios is given a name (for example, $\frac{b}{c}$ is called sine θ, $\frac{a}{c}$ is called cosine θ, $\frac{b}{a}$ is called tangent θ and so on). Now we give the definition of these trigonometric ratios formally in the following. Later, we observe that this definition is independent of the triangle.

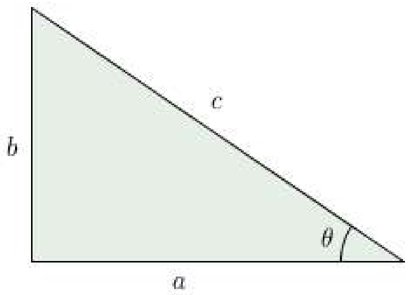


Fig. 6.4

6.1.1 Definition

Let θ be a real number and $0 \le \theta \le 2\pi$ and $r > 0$. Consider a rectangular coordinate system with OX, OY as axes. Draw a circle with centre O and radius r. Choose a point P on the circle such that the line OP makes an angle θ radians with $\overrightarrow{OX}$ (positive X-axis) measured in anti-clock wise direction (positive direction). See the figures below.

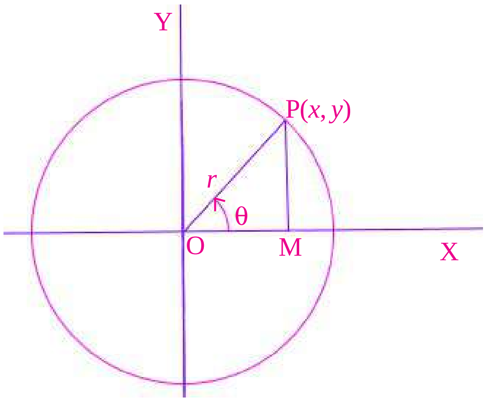


Fig. 6.5(i)

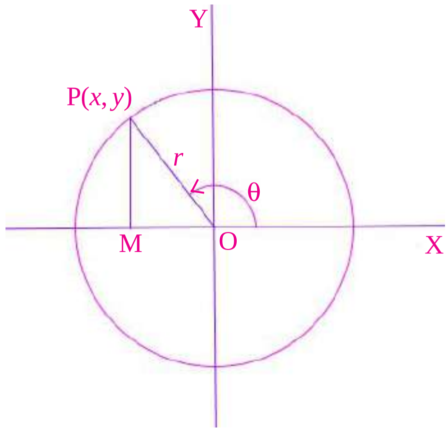


Fig. 6.5(ii)

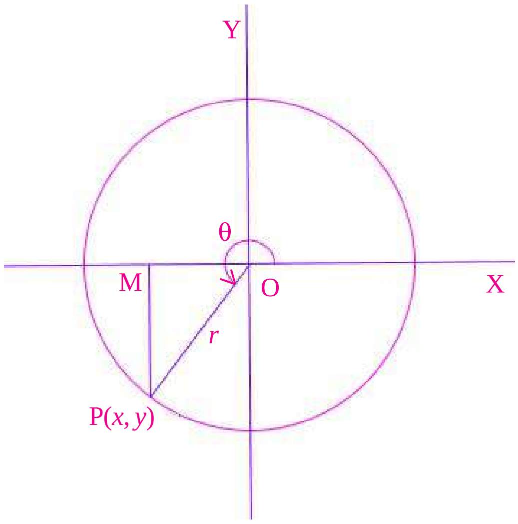


Fig. 6.5(iii)

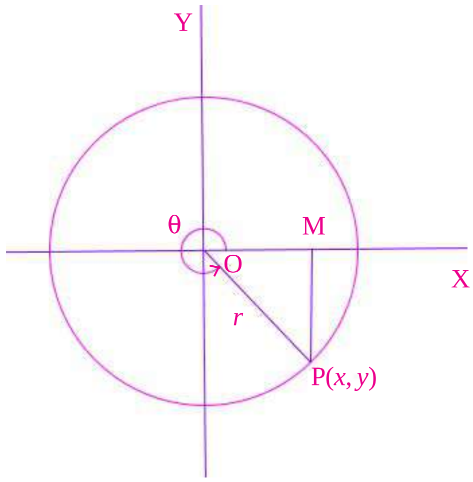


Fig. 6.5(iv)

Let (x, y) be the coordinates of P with reference to the coordinate axes OX and OY. First observe that

$$y = 0 \Leftrightarrow \theta = 0 \text{ or } \theta = \pi$$

$$x = 0 \Leftrightarrow \theta = \frac{\pi}{2} \text{ or } \theta = \frac{3\pi}{2}.$$

We define the six trigonometric ratios of θ as follows :

$$\text{Sine of } \theta = \frac{y}{r}$$

$$\text{Cosine of } \theta = \frac{x}{r}$$

$$\text{Tangent of } \theta = \frac{y}{x} \text{ when } x \neq 0 \text{ or } \theta \notin \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$$

$$\text{Cotangent of } \theta = \frac{x}{y} \text{ when } y \neq 0 \text{ or } \theta \notin \{0, \pi\}$$

$$\text{Secant of } \theta = \frac{r}{x} \text{ when } x \neq 0 \text{ or } \theta \notin \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$$

$$\text{Cosecant of } \theta = \frac{r}{y} \text{ when } y \neq 0 \text{ or } \theta \notin \{0, \pi\}$$

First we observe that these trigonometric ratios are independent of the choice of r. Let us take two circles with radii r_1 and r_2 with $r_1 \neq r_2$. We can take $r_1 < r_2$.

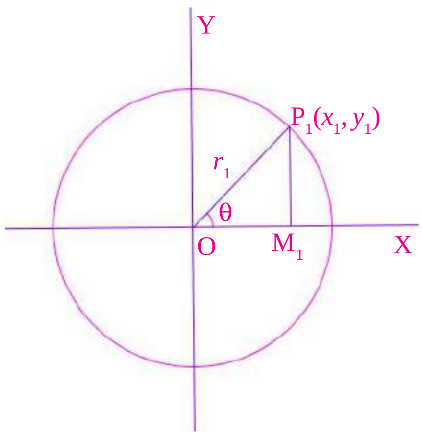


Fig. 6.6(i)

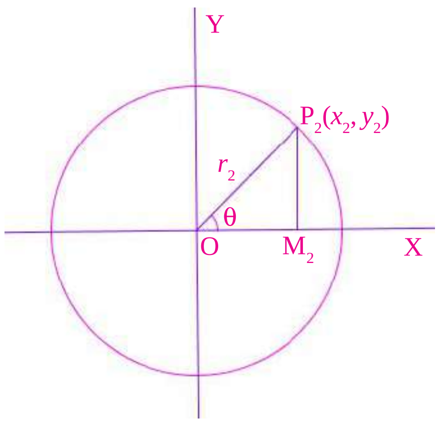


Fig. 6.6(ii)

Let $P_1(x_1, y_1)$ be a point on the circle with radius r_1 and centre 'O' such that angle $XOP_1 = \theta$ and $P_2(x_2, y_2)$ be a point on the circle with radius r_2 and centre 'O' such that angle $XOP_2 = \theta$. Draw perpendiculars P_1M_1 and P_2M_2 from P_1 and P_2 respectively to OX. Then the triangles OM_1P_1 and OM_2P_2 are similar (two right angled triangles with same angles $\theta, \frac{\pi}{2} - \theta, \frac{\pi}{2}$). Hence the corresponding sides are proportional.

Thus we get $\dfrac{OM_1}{OM_2} = \dfrac{M_1P_1}{M_2P_2} = \dfrac{OP_1}{OP_2}$ or $\dfrac{x_1}{x_2} = \dfrac{y_1}{y_2} = \dfrac{r_1}{r_2}$.

Hence we get $\dfrac{y_1}{r_1} = \dfrac{y_2}{r_2}$; $\dfrac{x_1}{r_1} = \dfrac{x_2}{r_2}$; $\dfrac{y_1}{x_1} = \dfrac{y_2}{x_2}$ so on.

Thus the above definitions of the six trigonometric ratios of θ are independent of the choice of r (or the size of the triangle).

The six trigonometric ratios of θ defined above are briefly written as $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\operatorname{cosec}\theta$ respectively. From these definitions we can observe the following :

6.1.2 Note

1. $\sin\theta = 0 \Leftrightarrow y = 0 \Leftrightarrow \theta = 0$ or $\theta = \pi$.

2. $\cos\theta = 0 \Leftrightarrow x = 0 \Leftrightarrow \theta = \dfrac{\pi}{2}$ or $\theta = \dfrac{3\pi}{2}$.

3. If $x \neq 0$, then $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}$.

4. If $y \neq 0$, then $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ and $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$.

5. In triangle OMP, from Pythagoras theorem, $x^2 + y^2 = r^2$.

So $\left(\dfrac{x}{r}\right)^2 + \left(\dfrac{y}{r}\right)^2 = 1$.

Hence $$\cos^2\theta + \sin^2\theta = 1$$

If $\theta \notin \left\{\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right\}$, then $\cos\theta \neq 0$ and hence on dividing both sides by $\cos^2\theta$, we get

$$1 + \tan^2\theta = \sec^2\theta$$

Similarly, if $\theta \notin \{0, \pi\}$, then we get

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

6. From the definitions of the six trigonometric ratios given in 6.1.1 we can make the following important observations.

(i) If P(x, y) is in the first quadrant (but not on the coordinate axes), that is, if $0 < \theta < \dfrac{\pi}{2}$, then $x > 0$ and $y > 0$. Hence all the six trigonometric ratios are positive. (see Fig. 6.5(i)).

(ii) If P(x, y) lies in the second quadrant (but not on the coordinate axes), that is, if $\frac{\pi}{2} < \theta < \pi$, then $x < 0$ and $y > 0$. Hence $\sin\theta$ and consequently $\operatorname{cosec}\theta$ are positive and other trigonometric ratios are negative (see Fig. 6.5(ii)).

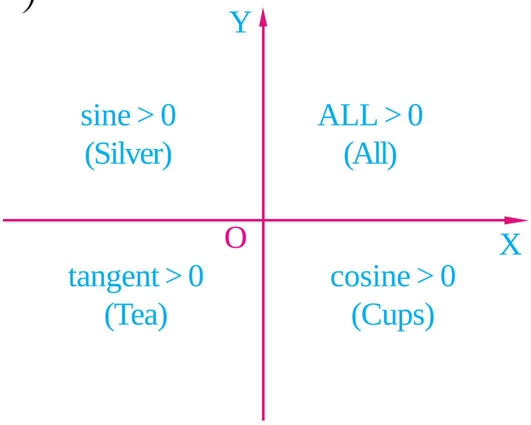
(iii) If P(x, y) lies in third quadrant $\left(\pi < \theta < \frac{3\pi}{2}\right)$, then $\tan\theta$ and $\cot\theta$ are positive and others are negative (see Fig. 6.5(iii)).

(iv) If P(x, y) lies in fourth quadrant $\left(\frac{3\pi}{2} < \theta < 2\pi\right)$, then $\cos\theta$ and $\sec\theta$ are positive and others are negative (see Fig. 6.5(iv)).

In all the above 4 cases P does not lie on coordinate axes, that is,

$$\theta \in [0, 2\pi] \setminus \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right\}.$$

The above facts can be easily remembered by the adjacent diagram.



The trigonometric ratios which are positive in various quadrants can also be remembered as follows.

I	II	III	IV
All	Silver	Tea	Cups
(all	sine	tan	cos)

It is enough to know about the properties of sine, cosine and tangent of the angle θ in the four quadrants as the remaining trigonometric ratios are their reciprocals only.

So far, we have defined trigonmetric ratios only when $\theta \in [0, 2\pi]$.

That is we have taken subsets of $[0, 2\pi]$ as domains for these trigonometric ratios.

Now we shall extend the domains of definitions of the trigonometric functions $\sin\theta$, $\cos\theta \cdots$ to the whole real number system.

6.1.3 Definition

For any real number x, let n be the largest integer such that $2n\pi \le x$. (That is, n is the integral part of $\frac{x}{2\pi}$). Write $\theta = x - 2n\pi$. Then $0 \le \theta \le 2\pi$. We define

$$\sin x = \sin\theta = \sin(x - 2n\pi)$$
$$\text{and} \quad \cos x = \cos\theta = \cos(x - 2n\pi)$$

Note that, for any $0 \le \theta < 2\pi$ and for any integer n, we have

$$\sin(2n\pi + \theta) = \sin\theta$$

$$\text{and } \cos(2n\pi + \theta) = \cos\theta \text{ etc.}$$

For example, $\sin\frac{\pi}{6} = \sin\frac{13\pi}{6}$, $\cos\left(\frac{\pi}{4}\right) = \cos\frac{17\pi}{4}$ etc.

If the angle θ $(0 \le \theta < 2\pi)$ is measured in anti-clockwise direction (starting from the initial side OX), it is defined as **positive** angle and if the same angle θ is measured in clockwise direction, it is defined as **negative** angle and it is identified with $-\theta$. (see the Fig. 6.7.

The trigonometric ratios of '$-\theta$' are defined as follows.

$$\sin(-\theta) = \sin(2\pi - \theta) = \frac{-y}{r}$$

$$= -\left(\frac{y}{r}\right) = -\sin\theta.$$

$$\cos(-\theta) = \cos(2\pi - \theta) = \frac{x}{r} = \cos\theta.$$

If $\theta \notin \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$, then $\tan(-\theta) = -\tan\theta$.

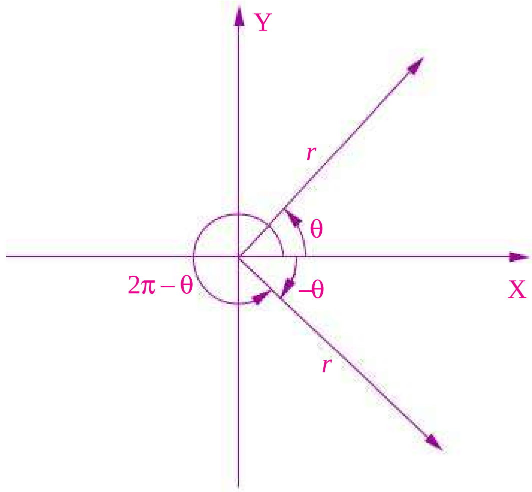


Fig. 6.7

6.1.4 Definition

*The angles $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ have their terminal side along either X-axis or Y-axis. Hence these angles are called "**Quadrant angles**".*

We have learnt the values of the trigonometric ratios of the angles $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ in earlier classes. The values of the trigonometric ratios of these angles and the quadrant angles are given in the following table.

Table 6.1

Angle (θ) Trigonometric ratio	$0^C(0^0)$	$\frac{\pi}{6}^C(30^0)$	$\frac{\pi}{4}^C(45^0)$	$\frac{\pi}{3}^C(60^0)$	$\frac{\pi}{2}^C(90^0)$	$\pi^C(180^0)$	$\frac{3\pi}{2}^C(270^0)$	$2\pi^C(360^0)$
$\sin\theta$	$0=\sqrt{\frac{0}{4}}$	$\frac{1}{2}=\sqrt{\frac{1}{4}}$	$\frac{1}{\sqrt{2}}=\sqrt{\frac{2}{4}}$	$\frac{\sqrt{3}}{2}=\sqrt{\frac{3}{4}}$	$1=\sqrt{\frac{4}{4}}$	0	–1	0
$\cos\theta$	$1=\sqrt{\frac{4}{4}}$	$\frac{\sqrt{3}}{2}=\sqrt{\frac{3}{4}}$	$\frac{1}{\sqrt{2}}=\sqrt{\frac{2}{4}}$	$\frac{1}{2}=\sqrt{\frac{1}{4}}$	$0=\sqrt{\frac{0}{4}}$	–1	0	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined	0	Not defined	0

We can put any real number α in the form $\alpha = \frac{n\pi}{2} + \theta$ as well as $\alpha = \frac{m\pi}{2} - \theta$ for some $\theta \in \left[0, \frac{\pi}{2}\right]$ and for some integers n and m. Hence, in the following table we give the change that occurs in a trigonometric ratio when applied on the angles in the form $\frac{n\pi}{2} \pm \theta \left(\theta \in \left(0, \frac{\pi}{2}\right)\right)$. When $\theta = 0$ or $\frac{\pi}{2}$, some of the trigonometric ratios are undefined and hence we give their values separately.

Table 6.2

Angle (α) Trigonometric ratio	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$
$n\pi - \theta$	$(-1)^{n+1}\sin\theta$	$(-1)^n\cos\theta$	$-\tan\theta$
$n\pi + \theta$	$(-1)^n\sin\theta$	$(-1)^n\cos\theta$	$\tan\theta$
$(2n+1)\frac{\pi}{2} - \theta$	$(-1)^n\cos\theta$	$(-1)^n\sin\theta$	$\cot\theta$
$(2n+1)\frac{\pi}{2} + \theta$	$(-1)^n\cos\theta$	$(-1)^{n+1}\sin\theta$	$-\cot\theta$

The following useful observations can be made from the above table.

1. If a trigonometric ratio is applied on $n\frac{\pi}{2} \pm \theta \ (n \in \mathbf{Z})$, then
   (i) When n is even, there is no change in the trigonometric ratio (sign may be + or –).
   (ii) When n is odd, the change in the trigonometric ratio (sign may be + or –) is as indicated below
   sine $\leftrightarrow$ cosine; tangent $\leftrightarrow$ cotangent; secant $\leftrightarrow$ cosecant .
2. Whether we get + or – sign in the answer, should be decided by taking into consideration the quadrant in which the angle $n\frac{\pi}{2} \pm \theta$ lies.

Note: We usually take angles in radians. In case, we take an angle θ in degrees we write θ^0. If nothing is mentioned we assume that the angle is given in radians.

Example : Find the values of

(i) $\sin 210^0$ (ii) $\cos 585^0$ (iii) $\tan 480^0$

(iv) $\sec 510^0$ (v) $\operatorname{cosec} 750^0$ (vi) $\cot 765^0$

Solution

(i) $\sin 210^0 = \sin\left(180^0 + 30^0\right) = -\sin 30^0 = -\frac{1}{2}$.

(or) $\sin 210^0 = \sin\left(270^0 - 60^0\right) = -\cos 60^0 = -\frac{1}{2}$.

(ii) $\cos 585^0 = \cos(540^0 + 45^0) = \cos(3(180^0) + 45^0) = -\cos 45^0 = -\dfrac{1}{\sqrt{2}}$.

(iii) $\tan 480^0 = \tan(450^0 + 30^0) = \tan(5(90^0) + 30^0) = -\cot 30^0 = -\sqrt{3}$.

(iv) $\sec 510^0 = \sec(450^0 + 60^0) = \sec(5(90^0) + 60^0) = -\operatorname{cosec} 60^0 = -\dfrac{2}{\sqrt{3}}$.

(v) $\operatorname{cosec} 750^0 = \operatorname{cosec}(2(360^0) + 30^0) = \operatorname{cosec} 30^0 = 2$.

(vi) $\cot 765^0 = \cot(2(360^0) + 45^0) = \cot 45^0 = 1$.

Now we list the changes that occur in various trigonometric ratios when applied on angles of the form $\dfrac{n\pi}{2} \pm \theta$, where $\theta = 0$ or $\pi/2$ and $n \in \mathbf{Z}$.

(i) $\sin n\pi = 0 = \tan n\pi$ and hence $\operatorname{cosec} n\pi$, $\cot n\pi$ are undefined.

(ii) $\cos n\pi = (-1)^n = \sec n\pi$.

(iii) $\cos(2n+1)\dfrac{\pi}{2} = 0 = \cot(2n+1)\dfrac{\pi}{2}$ and hence $\sec(2n+1)\dfrac{\pi}{2}$, $\tan(2n+1)\dfrac{\pi}{2}$ are undefined.

(iv) $\sin(2n+1)\dfrac{\pi}{2} = (-1)^n = \operatorname{cosec}(2n+1)\dfrac{\pi}{2}$.

6.1.5 Definition

If θ is any angle then $\dfrac{\pi}{2} - \theta$ is called its ***complement*** *and $\pi - \theta$ is called its* ***supplement.***

In other words, two angles θ, ϕ are said to be ***complementary angles*** *if $\theta + \phi = \dfrac{\pi}{2}$ and* ***supplementary angles*** *if $\theta + \phi = \pi$. For example, the angles $\dfrac{\pi}{6}, \dfrac{\pi}{3}$ are complementary angles and $\dfrac{\pi}{6}, \dfrac{5\pi}{6}$ are supplementary angles.*

6.1.6 Solved Problems

1. Problem: *Find the values of*

(i) $\sin \dfrac{5\pi}{3}$ (ii) $\tan(855^0)$ (iii) $\sec\left(13\dfrac{\pi}{3}\right)$

Solution

(i) $\sin \dfrac{5\pi}{3} = \sin\left(2\pi - \dfrac{\pi}{3}\right) = -\sin\dfrac{\pi}{3} = -\dfrac{\sqrt{3}}{2}$.

(ii) $\tan\left(855^0\right) = \tan\left(900^0 - 45^0\right) = \tan\left(5\left(180^0\right) - 45^0\right) = -\tan 45^0 = -1.$

(iii) $\sec\left(13\dfrac{\pi}{3}\right) = \sec\left(4\pi + \dfrac{\pi}{3}\right) = \sec\left(2\left(2\pi\right) + \dfrac{\pi}{3}\right) = \sec\dfrac{\pi}{3} = 2.$

2. Problem: *Simplify*

(i) $\cot\left(\theta - \dfrac{13\pi}{2}\right)$ (ii) $\tan\left(-23\dfrac{\pi}{3}\right)$

Solution

(i) $\cot\left(\theta - \dfrac{13\pi}{2}\right) = \cot\left[-\left(\dfrac{13\pi}{2} - \theta\right)\right] = -\cot\left(\dfrac{13\pi}{2} - \theta\right)$

$= -\cot\left[6\pi + \left(\dfrac{\pi}{2} - \theta\right)\right] = -\cot\left(\dfrac{\pi}{2} - \theta\right)$

$= -\tan\theta.$

(ii) $\tan\left(-23\dfrac{\pi}{3}\right) = -\tan\left(23\dfrac{\pi}{3}\right)$

$= -\tan\left[6\pi + \dfrac{5\pi}{3}\right]$

$= -\tan\left(\dfrac{5\pi}{3}\right) = -\tan\left(2\pi - \dfrac{\pi}{3}\right)$

$= \tan\dfrac{\pi}{3} = \sqrt{3}.$

3. Problem: *Find the value of* $\sin^2\dfrac{\pi}{10} + \sin^2\dfrac{4\pi}{10} + \sin^2\dfrac{6\pi}{10} + \sin^2\dfrac{9\pi}{10}.$

Solution: $\sin^2\dfrac{\pi}{10} + \sin^2\dfrac{4\pi}{10} + \sin^2\dfrac{6\pi}{10} + \sin^2\dfrac{9\pi}{10}$

$= \sin^2\left(\dfrac{\pi}{10}\right) + \sin^2\left(\dfrac{\pi}{2} - \dfrac{\pi}{10}\right) + \sin^2\left(\dfrac{\pi}{2} + \dfrac{\pi}{10}\right) + \sin^2\left(\pi - \dfrac{\pi}{10}\right)$

$= \sin^2\dfrac{\pi}{10} + \cos^2\dfrac{\pi}{10} + \cos^2\dfrac{\pi}{10} + \sin^2\dfrac{\pi}{10}$

$= 2$ (since, for any angle θ, $\sin^2\theta + \cos^2\theta = 1$).

4. Problem: *If* $\sin\theta = \dfrac{4}{5}$ *and* θ *is not in the first quadrant, find the value of* $\cos\theta$.

Solution: $\sin\theta$ is positive and θ is not in the first quadrant. Hence θ is in the second quadrant and therefore $\cos\theta < 0$.

We have

$$\cos^2\theta + \sin^2\theta = 1 \quad \text{(see note 6.1.2 (5))}$$

$$\Rightarrow \cos^2\theta = 1 - \sin^2\theta = 1 - \frac{16}{25} = \frac{9}{25}$$

$$\Rightarrow \cos\theta = \pm 3/5$$

$$\therefore \cos\theta = -3/5 \ (\text{since } \cos\theta < 0).$$

5. Problem: *If* $\sec\theta + \tan\theta = \frac{2}{3}$, *find the value of* $\sin\theta$ *and determine the quadrant in which* θ *lies.*

Solution: We know that $\sec^2\theta - \tan^2\theta = 1$.

$$\text{So } \sec\theta - \tan\theta = \frac{1}{\sec\theta + \tan\theta} = \frac{3}{2}$$

$$\therefore (\sec\theta + \tan\theta) + (\sec\theta - \tan\theta) = \frac{2}{3} + \frac{3}{2}$$

$$\Rightarrow 2\sec\theta = \frac{13}{6} \Rightarrow \sec\theta = \frac{13}{12}.$$

Again, $(\sec\theta + \tan\theta) - (\sec\theta - \tan\theta) = \frac{2}{3} - \frac{3}{2} = \frac{-5}{6}$

$$\Rightarrow 2\tan\theta = \frac{-5}{6} \Rightarrow \tan\theta = \frac{-5}{12} \text{ and } \sin\theta = \frac{-5}{12} \div \frac{13}{12} = \frac{-5}{13}.$$

Since $\sec\theta$ is +ve and $\tan\theta$ is –ve, θ lies in the IV quadrant.

6. Problem: *Prove that* $\cot\frac{\pi}{16} \cdot \cot\frac{2\pi}{16} \cdot \cot\frac{3\pi}{16} \cdots \cot\frac{7\pi}{16} = 1$.

Solution: $\cot\frac{\pi}{16} \cdot \cot\frac{2\pi}{16} \cdot \cot\frac{3\pi}{16} \cdots \cot\frac{7\pi}{16}$

$$= \left(\cot\frac{\pi}{16} \cdot \cot\frac{7\pi}{16}\right) \cdot \left(\cot\frac{2\pi}{16} \cdot \cot\frac{6\pi}{16}\right) \cdot \cot\left(\frac{3\pi}{16} \cdot \cot\frac{5\pi}{16}\right) \cdot \cot\frac{4\pi}{16}$$

$$= \left(\cot\frac{\pi}{16} \cdot \cot\left(\frac{\pi}{2} - \frac{\pi}{16}\right)\right) \cdot \left(\cot\frac{2\pi}{16} \cdot \cot\left(\frac{\pi}{2} - \frac{2\pi}{16}\right)\right).$$

$$\left(\cot\frac{3\pi}{16} \cdot \cot\left(\frac{\pi}{2} - \frac{3\pi}{16}\right)\right) \cdot \cot\frac{\pi}{4}$$

$$= \left(\cot\frac{\pi}{16} \cdot \tan\frac{\pi}{16}\right) \cdot \left(\cot\frac{2\pi}{16} \cdot \tan\frac{2\pi}{16}\right)\left(\cot\frac{3\pi}{16} \cdot \tan\frac{3\pi}{16}\right) \cdot 1$$

$$= 1 . 1 . 1 . 1 = 1.$$

7. Problem: *If* $3\sin\theta + 4\cos\theta = 5$, *then find the value of* $4\sin\theta - 3\cos\theta$.

Solution: Given that $3\sin\theta + 4\cos\theta = 5$

write $4\sin\theta - 3\cos\theta = a$

Squaring and adding we get $(3\sin\theta + 4\cos\theta)^2 + (4\sin\theta - 3\cos\theta)^2 = 5^2 + a^2$

$$\Rightarrow a^2 + 25 = 9\sin^2\theta + 16\cos^2\theta + 24\sin\theta\cos\theta + 16\sin^2\theta + 9\cos^2\theta - 24\sin\theta\cos\theta$$

$$= 25\sin^2\theta + 25\cos^2\theta = 25.$$

$$\Rightarrow a^2 = 0 \Rightarrow a = 0.$$

$\therefore 4\sin\theta - 3\cos\theta = 0.$

8. Problem: *If* $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, *prove that* $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$.

Solution: Given $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, then $\left(\sqrt{2} - 1\right)\cos\theta = \sin\theta$.

On multiplying both sides by $\left(\sqrt{2} + 1\right)$, we get

$$\left(\sqrt{2} + 1\right)\left(\sqrt{2} - 1\right)\cos\theta = \left(\sqrt{2} + 1\right)\sin\theta$$

$$\Rightarrow \cos\theta = \sqrt{2}\sin\theta + \sin\theta$$

$$\Rightarrow \cos\theta - \sin\theta = \sqrt{2}\sin\theta.$$

9. Problem: *Find the value of* $2\left(\sin^6\theta + \cos^6\theta\right) - 3\left(\sin^4\theta + \cos^4\theta\right)$.

Solution: $2\left(\sin^6\theta + \cos^6\theta\right) - 3\left(\sin^4\theta + \cos^4\theta\right)$

$$= 2\left\{\left(\sin^2\theta\right)^3 + \left(\cos^2\theta\right)^3\right\} - 3\left\{\left(\sin^2\theta\right)^2 + \left(\cos^2\theta\right)^2\right\}$$

$$= 2\left\{\left(\sin^2\theta + \cos^2\theta\right)^3 - 3\sin^2\theta\cos^2\theta\left(\sin^2\theta + \cos^2\theta\right)\right\}$$

$$-3\left\{\left(\sin^2\theta + \cos^2\theta\right)^2 - 2\sin^2\theta\cos^2\theta\right\}$$

$$= 2\left(1 - 3\sin^2\theta\cos^2\theta\right) - 3\left(1 - 2\sin^2\theta\cos^2\theta\right)$$

$$= -1.$$

10. Problem: *Prove that* $(\tan\theta + \cot\theta)^2 = \sec^2\theta + \mathrm{cosec}^2\theta = \sec^2\theta . \mathrm{cosec}^2\theta$.

Solution: $(\tan\theta + \cot\theta)^2 = \tan^2\theta + \cot^2\theta + 2\tan\theta\cot\theta$

$$= \tan^2\theta + \cot^2\theta + 2$$

$$= \left(1 + \tan^2\theta\right) + \left(1 + \cot^2\theta\right)$$

$$= \sec^2\theta + \mathrm{cosec}^2\theta.$$

Again, $\sec^2\theta + \mathrm{cosec}^2\theta = \dfrac{1}{\cos^2\theta} + \dfrac{1}{\sin^2\theta}$

$$= \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta . \sin^2\theta} = \frac{1}{\cos^2\theta . \sin^2\theta} = \sec^2\theta . \mathrm{cosec}^2\theta.$$

11. Problem: *If* $\cos\theta > 0$, $\tan\theta + \sin\theta = m$ *and* $\tan\theta - \sin\theta = n$, *then show that* $m^2 - n^2 = 4\sqrt{mn}$.

Solution: Given that $m = \tan\theta + \sin\theta$ *and* $n = \tan\theta - \sin\theta$.

By adding, we get $m + n = 2\tan\theta$.

By subtracting, we get $m - n = 2\sin\theta$.

On multiplying these two equations, we get $m^2 - n^2 = 4\tan\theta\,\sin\theta$

$$= 4\sqrt{\tan^2\theta\,.\,\sin^2\theta} = 4\sqrt{\tan^2\theta\left(1 - \cos^2\theta\right)} \quad (\text{since } \cos\theta > 0)$$

$$= 4\sqrt{\tan^2\theta - \sin^2\theta} = 4\sqrt{mn}.$$

12. Problem: *If* $\tan 20^0 = \lambda$, *then show that* $\dfrac{\tan 160^0 - \tan 110^0}{1 + \tan 160^0 . \tan 110^0} = \dfrac{1 - \lambda^2}{2\lambda}$.

Solution: L.H.S. $= \dfrac{\tan 160^0 - \tan 110^0}{1 + \tan 160^0 . \tan 110^0}$

$$= \frac{\tan\left(180^0 - 20^0\right) - \tan\left(90^0 + 20^0\right)}{1 + \tan\left(180^0 - 20^0\right) . \tan\left(90^0 + 20^0\right)}$$

$$= \frac{-\tan 20^0 + \cot 20^0}{1 + \left(-\tan 20^0\right)\left(-\cot 20^0\right)} = \frac{-\lambda + \frac{1}{\lambda}}{1 + 1} = \frac{1 - \lambda^2}{2\lambda} = \text{R.H.S.}$$

Exercise 6(a)

I.1. Convert the following into simplest form

(i) $\tan(\theta - 14\pi)$ (ii) $\cot\left(\dfrac{21\pi}{2} - \theta\right)$

(iii) $\operatorname{cosec}(5\pi + \theta)$ (iv) $\sec(4\pi - \theta)$

2. Find the value of each of the following

(i) $\sin(-405^0)$ (ii) $\cos\left(-\dfrac{7\pi}{2}\right)$

(iii) $\sec(2100^0)$ (iv) $\cot(-315^0)$

3. Evaluate

(i) $\cos^2 45^0 + \cos^2 135^0 + \cos^2 225^0 + \cos^2 315^0$

(ii) $\sin^2\dfrac{2\pi}{3} + \cos^2\dfrac{5\pi}{6} - \tan^2\dfrac{3\pi}{4}$

(iii) $\cos 225^0 - \sin 225^0 + \tan 495^0 - \cot 495^0$

(iv) $(\cos\theta - \sin\theta)$ if (a) $\theta = \dfrac{7\pi}{4}$ (b) $\theta = \dfrac{11\pi}{3}$

4. (i) If $\sin\theta = -\dfrac{1}{3}$ and θ does not lie in the third quadrant, find the values of

(a) $\cos\theta$ (b) $\cot\theta$

(ii) If $\cos\theta = t$ $(0 < t < 1)$ and θ does not lie in the first quadrant, find the values of

(a) $\sin\theta$ (b) $\tan\theta$.

(iii) Find the value of $\sin 330^0 . \cos 120^0 + \cos 210^0 . \sin 300^0$

(iv) If $\text{cosec}\,\theta + \cot\theta = \dfrac{1}{3}$, find $\cos\theta$ and determine the quadrant in which θ lies.

5. (i) If $\sin\alpha + \text{cosec}\,\alpha = 2$, find the value of $\sin^n\alpha + \text{cosec}^n\alpha$, $n \in \mathbf{Z}$.

(ii) If $\sec\theta + \tan\theta = 5$, find the quadrant in which θ lies and find the value of $\sin\theta$.

II.

1. Prove that

(i) $$\frac{\cos(\pi - A) . \cot\left(\frac{\pi}{2} + A\right)\cos(-A)}{\tan(\pi + A)\tan\left(\frac{3\pi}{2} + A\right)\sin(2\pi - A)} = \cos A .$$

(ii) $$\frac{\sin(3\pi - A)\cos\left(A - \frac{\pi}{2}\right)\tan\left(\frac{3\pi}{2} - A\right)}{\text{cosec}\left(\frac{13\pi}{2} + A\right)\sec(3\pi + A)\cot\left(A - \frac{\pi}{2}\right)} = \cos^4 A .$$

(iii) $\sin 780^0 \sin 480^0 + \cos 240^0 . \cos 300^0 = \dfrac{1}{2}$.

(iv) $$\frac{\sin 150^0 - 5\cos 300^0 + 7\tan 225^0}{\tan 135^0 + 3\sin 210^0} = -2 .$$

(v) $$\cot\left(\frac{\pi}{20}\right).\cot\left(\frac{3\pi}{20}\right).\cot\left(\frac{5\pi}{20}\right).\cot\left(\frac{7\pi}{20}\right).\cot\left(\frac{9\pi}{20}\right) = 1 .$$

2. (i) Simplify

$$\frac{\sin\left(-\frac{11\pi}{3}\right)\tan\left(\frac{35\pi}{6}\right)\sec\left(-\frac{7\pi}{3}\right)}{\cos\left(\frac{5\pi}{4}\right)\text{cosec}\left(\frac{7\pi}{4}\right)\cos\left(\frac{17\pi}{6}\right)} .$$

(ii) If $\tan 20^0 = p$, prove that $\dfrac{\tan 610^0 + \tan 700^0}{\tan 560^0 - \tan 470^0} = \dfrac{1 - p^2}{1 + p^2}$.

(iii) If α, β are complementary angles such that $b\sin\alpha = a$, then find the value of $(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$.

3. (i) If $\cos A = \cos B = -\frac{1}{2}$ and A does not lie in the second quadrant and B does not lie in the third quadrant, then find the value of $\frac{4\sin B - 3\tan A}{\tan B + \sin A}$.

(ii) If $8\tan A = -15$ and $25\sin B = -7$ and neither A nor B is in the fourth quadrant, then show that $\sin A\cos B + \cos A\sin B = \frac{-304}{425}$.

(iii) If A, B, C, D are angles of a cyclic quadrilateral, then prove that

(a) $\sin A - \sin C = \sin D - \sin B$ and

(b) $\cos A + \cos B + \cos C + \cos D = 0$.

4. (i) If $a\cos\theta - b\sin\theta = c$, then show that $a\sin\theta + b\cos\theta = \pm\sqrt{a^2+b^2-c^2}$.

(ii) If $3\sin A + 5\cos A = 5$, then show that $5\sin A - 3\cos A = \pm 3$.

(iii) If $\tan^2\theta = (1-e^2)$, show that $\sec\theta + \tan^3\theta \cdot \operatorname{cosec}\theta = (2-e^2)^{3/2}$.

III.

1. Prove the following:

(i) $\frac{(\tan\theta + \sec\theta - 1)}{(\tan\theta - \sec\theta + 1)} = \frac{1+\sin\theta}{\cos\theta}$.

(ii) $(1 + \cot\theta - \operatorname{cosec}\theta)(1 + \tan\theta + \sec\theta) = 2$.

(iii) $3(\sin\theta - \cos\theta)^4 + 6(\sin\theta + \cos\theta)^2 + 4(\sin^6\theta + \cos^6\theta) = 13$.

2. Prove that

(i) $(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 - (\tan^2\theta + \cot^2\theta) = 7$.

(ii) $\cos^4\alpha + 2\cos^2\alpha\left(1 - \frac{1}{\sec^2\alpha}\right) = (1 - \sin^4\alpha)$.

(iii) $\frac{(1+\sin\theta - \cos\theta)^2}{(1+\sin\theta+\cos\theta)^2} = \frac{1-\cos\theta}{1+\cos\theta}$.

(iv) If $\frac{2\sin\theta}{(1+\cos\theta+\sin\theta)} = x$, then find the value of $\frac{(1-\cos\theta+\sin\theta)}{(1+\sin\theta)}$.

3. Eliminate θ from the following:

(i) $x = a\cos^3\theta$; $y = b\sin^3\theta$

(ii) $x = a\cos^4\theta$; $y = b\sin^4\theta$

(iii) $x = a(\sec\theta + \tan\theta)$; $y = b(\sec\theta - \tan\theta)$

(iv) $x = \cot\theta + \tan\theta$; $y = \sec\theta - \cos\theta$

6.1.7 Definition (Domain, Range and Graph of a function)

Let A, B *be two sets and* $f : A \to B$ *be a function. Let us recall that* A *is called the* ***domain*** *of* f *and* B *is called the* ***codomain*** *of* f *and that the set* $\{f(x) \mid x \in A\}$ *is called the* ***range*** *of* f *(range of* f *is always a subset of the codomain of* f*). The subset* $\{(x, f(x)) \mid x \in A\}$ *of* $A \times B$ *is called the* ***graph*** *of the function* f.

6.1.8 Definition (Periodic function, Period)

Let $E \subseteq \mathbf{R}$ *and* $f : E \to \mathbf{R}$ *be a function. Then* f *is called a* ***'Periodic function'*** *if there exists a positive real number 'p' such that*

(i) $(x + p) \in E$ *for all* $x \in E$ *and*

(ii) $f(x + p) = f(x)$ *for all* $x \in E$.

*If such a positive real number 'p' exists, then it is called '****a period****' of f.*

It can be easily observed that if $f : E \to \mathbf{R}$ *is a periodic function and 'p' is a period of f, then for any positive integer n, we get*

(i) $(x + np) \in E$ *for all* $x \in E$ *and* (ii) $f(x + np) = f(x)$ *for all* $x \in E$.

Hence 'np' is also a period of f.

If $f : E \to \mathbf{R}$ *is a periodic function and if there exists smallest positive real number p such that* $f(x + p) = f(x)$ *for all* $x \in E$ *then 'p' is called* ***'the period'*** *of f.*

It can be noted that a function f may be periodic without having 'the period'. For example, if we take any constant function $f : E \to \mathbf{R}$. (That is $f(x) = k$ for all $x \in \mathbf{R}$), then any positive real number is **a period** of f but f does not have **the period.**

For any real number θ, we have observed that θ and $2\pi + \theta$ have same trigonometric ratios (see definition 6.1.3) and hence all trigonometric functions $(f(x) = \sin x,\ f(x) = \cos x$ *etc.*$)$ are periodic. Now we find **the periods** of the trigonometric functions.

6.1.9 Theorem

The sine function is periodic and 2π *is* ***the period****.*

Proof : Define $f(x) = \sin x$ for all $x \in \mathbf{R}$. *Then*

$$f(2\pi + x) = \sin(2\pi + x) = \sin x = f(x) \text{ for all } x \in \mathbf{R}.$$

Hence $f(x) = \sin x$ is a periodic function and 2π is **a period** of *f*.

Suppose $0 < k < 2\pi$ and k is a period of *f*. Then

$$f(x + k) = f(x) \text{ for all } x \in \mathbf{R}. \quad \text{... (1)}$$

In particular, $f(2\pi - k + k) = f(2\pi - k)$ (by taking $x = 2\pi - k$)

Thus $f(2\pi - k) = f(2\pi)$.

That is, $\sin(2\pi - k) = \sin 2\pi = 0$. ... (2)

Since $0 < k < 2\pi$, we get $0 < 2\pi - k < 2\pi$ and hence we get $2\pi - k = \pi$ from (2). Thus $k = \pi$. Now

$$1 = \sin\frac{\pi}{2} = \sin\left(\frac{\pi}{2} + k\right) \text{ (taking } \left(x = \frac{\pi}{2}\right) \text{ in (1))}$$

$$= \sin\frac{3\pi}{2} \quad (\text{since } k = \pi)$$

$$= -1, \text{ a contradiction.}$$

Thus 2π is the least positive real number such that $\sin(x + 2\pi) = \sin x$ for all $x \in \mathbf{R}$. Hence 2π is **the period** of $f(x) = \sin x$.

We can prove the following theorem similarly.

6.1.10 Theorem

1. *The function $f(x) = \cos x$ is periodic and 2π is the period.*
2. *The function $f(x) = \tan x$ is periodic and π is the period.*

While finding **the period** of a periodic function, the following points will be useful.

6.1.11 Note

1. If f is periodic, then so is λf, for any scalar λ.
2. Let $f : \mathbf{R} \to \mathbf{R}$ be a periodic function and p be a period of f. Let a, b, c be real constants such that $a \neq 0$. Then the function $g : \mathbf{R} \to \mathbf{R}$ defined by $g(x) = f(ax + b) + c$ for all $x \in \mathbf{R}$ is also periodic and $\frac{p}{|a|}$ is a period of g. Further, if p is the period of f, then $\frac{p}{|a|}$ is the period of g.
3. Let $f : A \to \mathbf{R}$, $g : B \to \mathbf{R}$ be two periodic functions, p_1 be a period of f and p_2 be a period of g. Let p be a common integral multiple of p_1 and p_2 and $C = A \cap B$.. Then, for any $x \in C$, we have $(x + p) \in C$. Now $f + g$, $f - g$ and fg are all periodic and p is **a period** of each of them. If $g(x) \neq 0$ for all $x \in C$, then $\frac{f}{g}$ is also periodic and p is **a period** of $\frac{f}{g}$.

6.1.12 Example

Find the period of the function f defined by

(i) $f(x) = \sin(5x + 3)$ for all $x \in \mathbf{R}$

(ii) $f(x) = x - [x]$ for all $x \in \mathbf{R}$, where $[x]$ = integral part of x.

Solution

(i) $f(x) = \sin(5x+3)$. We know that the function $g(x) = \sin x$ for all $x \in \mathbf{R}$, has the period 2π. Now $f(x) = g(5x+3)$. Hence by note 6.1.11(2) above, we get that f is periodic and the period of f is $\frac{2\pi}{|5|} = \frac{2\pi}{5}$.

(ii) $f(1+x) = 1 + x - [1+x] = 1 + x - \{1 + [x]\} = x - [x] = f(x)$

$\therefore$ f is a periodic function and 1 is a period of f, if $0 < \lambda < 1$.

Take $x = \frac{1-\lambda}{2}$. Then $0 < (\lambda + x) < 1$. Therefore, $[x] = 0 = [\lambda + x]$. Now

$f(\lambda + x) = \lambda + x - [\lambda + x] = \lambda + x$ and $f(x) = x - [x] = x$. Thus

$f(\lambda + x) \neq f(x)$.

Hence 1 is the period of f.

6.1.13 Variation of trigonometric ratios

(i) Variation of $\sin x$. (Fig 6.8)

As x increases from 0 to $\pi/2$, $\sin x$ increases from 0 to 1

As x increases from $\pi/2$ to π , $\sin x$ decreases from 1 to 0

As x increases from π to $3\pi/2$, $\sin x$ decreases from 0 to -1

As x increases from $3\pi/2$ to 2π, $\sin x$ increases from -1 to 0

(ii) Variation of $\cos x$. (Fig 6.9)

As x increases from 0 to $\pi/2$, $\cos x$ decreases from 1 to 0

As x increases from $\pi/2$ to π , $\cos x$ decreases from 0 to -1

As x increases from π to $3\pi/2$, $\cos x$ increases from -1 to 0

As x increases from $3\pi/2$ to 2π, $\cos x$ increases from 0 to 1

(iii) Variation of $\tan x$. (Fig 6.10)

As x increases from 0 to $\pi/2$, $\tan x$ increases from 0 to ∞

As x increases from $\pi/2$ to π , $\tan x$ increases from $-\infty$ to 0

As x increases from π to $3\pi/2$, $\tan x$ increases from 0 to ∞

As x increases from $3\pi/2$ to 2π, $\tan x$ increases from $-\infty$ to 0

Similarly, we can obtain the variations of $\operatorname{cosec} x$, $\sec x$ and $\cot x$. These variations can be easily understood from the graphs of these trigonometric functions given in 6.1.15.

6.1.14 Domain and Range of trigonometric functions

If we define $f(x) = \sin x$, then it is called a trigonometric function corresponding to the trigonometric ratio sine. Similarly, the trigonometric functions corresponding to the other trigonometric ratios can be defined. The domain and range of each of these trigonometric functions are given in Table 6.3.

Table 6.3

Trigonometric function	Domain (x)	Range (y)
$y = \sin x$	$\mathbf{R}$	[-1, 1]
$y = \cos x$	$\mathbf{R}$	[-1, 1]
$y = \tan x$	$\mathbf{R} \setminus \left\{(2n+1)\frac{\pi}{2} \,\middle\|\, n \in \mathbf{Z}\right\}$	$\mathbf{R}$
$y = \cot x$	$\mathbf{R} \setminus \{n\pi \mid n \in \mathbf{Z}\}$	$\mathbf{R}$
$y = \sec x$	$\mathbf{R} \setminus \left\{(2n+1)\frac{\pi}{2} \,\middle\|\, n \in \mathbf{Z}\right\}$	$(-\infty, -1] \cup [1, \infty)$
$y = \operatorname{cosec} x$	$\mathbf{R} \setminus \{n\pi \mid n \in \mathbf{Z}\}$	$(-\infty, -1] \cup [1, \infty)$

6.1.15 Graphs of trigonometric functions

We plot the graphs of the trigonometric functions by taking x in radians on X-axis and y on Y-axis. We first write the values of y corresponding to different values of x in a table and then by taking a suitable scale we plot these points in the coordinate plane and join these points by a smooth curve to get the graph.

1. Graph of $y = \sin x$

Table 6.4

x	$-\pi$	$-\pi/2$	0	$\pi/2$	π	$3\pi/2$	2π	$5\pi/2$	3π
$y = \sin x$	0	-1	0	1	0	-1	0	1	0

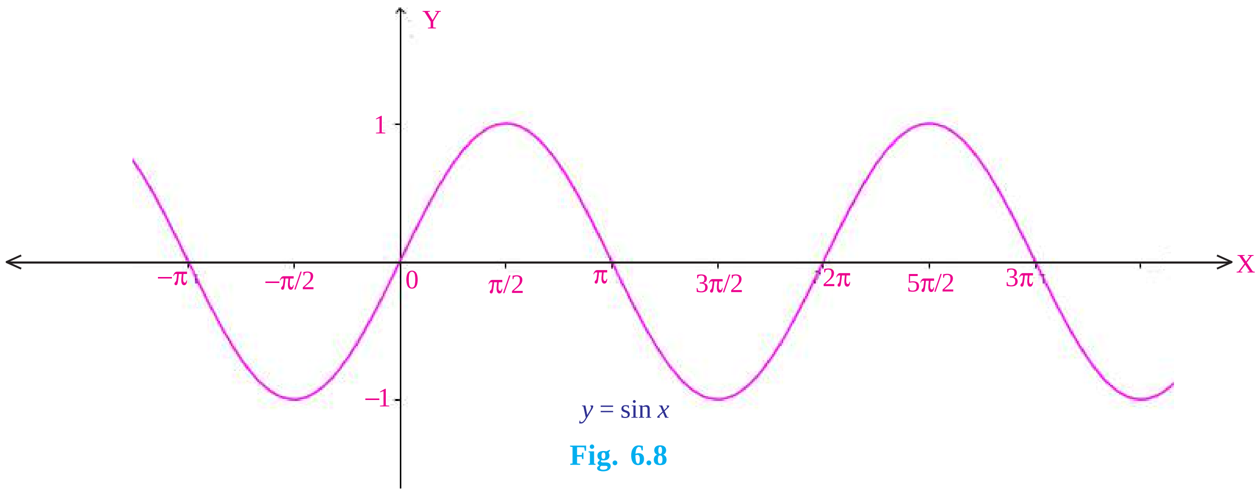


Fig. 6.8

2. Graph of $y = \cos x$

Table 6.5

x	$-\pi$	$-\pi/2$	0	$\pi/2$	π	$3\pi/2$	2π	$5\pi/2$	3π	$7\pi/2$
$y = \cos x$	-1	0	1	0	-1	0	1	0	-1	0

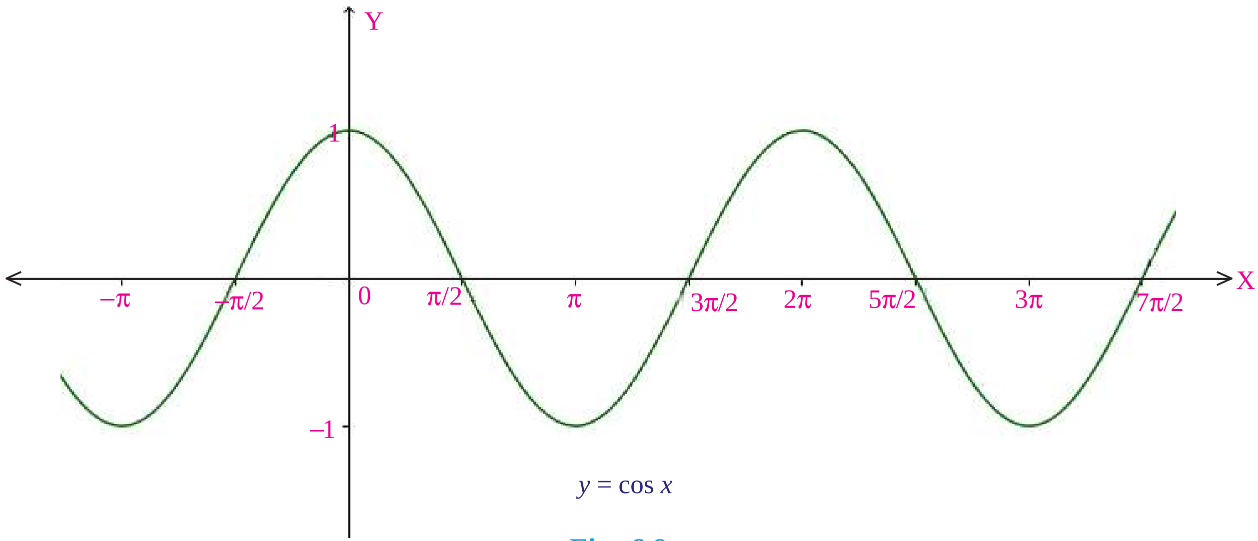


Fig. 6.9

3. Graph of $y = \tan x$

First observe that though $\tan x$ is not defined for $x = \pi/2$, $\tan x \to \infty$ as $x \to \pi/2$ in the interval $(0, \pi/2)$ and $\tan x \to -\infty$ as $x \to \frac{\pi}{2}$ in $(\pi/2, \pi)$. Similarly at $x = \frac{3\pi}{2}, \frac{5\pi}{2}$ also. We keep these points in mind while drawing the graph of $y = \tan x$.

Table 6.6

x	$-\pi$	$-\pi/2$	0	$\pi/2$	π	$3\pi/2$	2π	$5\pi/2$	3π
$y = \tan x$	0	not defined	0	not defined	0	not defined	0	not defined	0

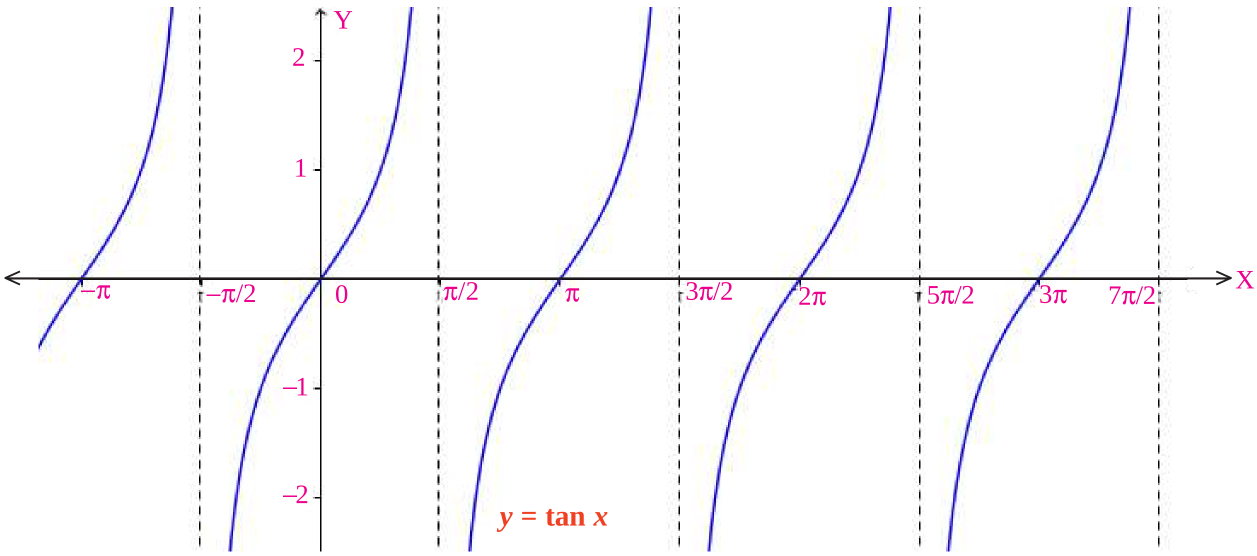


Fig. 6.10

4. Graph of $y = \cot x$

Before drawing the graph note that $\cot x$ is undefined at $-\pi, 0, \pi, 2\pi$ etc. since $\sin x = 0$ at these values of x. But $\cot x \to \infty$ *as* $x \to 0$ in $\left(0, \frac{\pi}{2}\right)$ and $\cot x \to -\infty$ *as* $x \to 0$ *in* $\left(-\frac{\pi}{2}, 0\right)$. Similarly for π, 2π etc. We keep these points in mind while drawing the graph of $y = \cot x$.

Table 6.7

x	$-\pi$	$-\pi/2$	0	$\pi/2$	π	$3\pi/2$	2π	$5\pi/2$	3π
$y = \cot x$	not defined	0	not defined	0	not defined	0	not defined	0	not defined

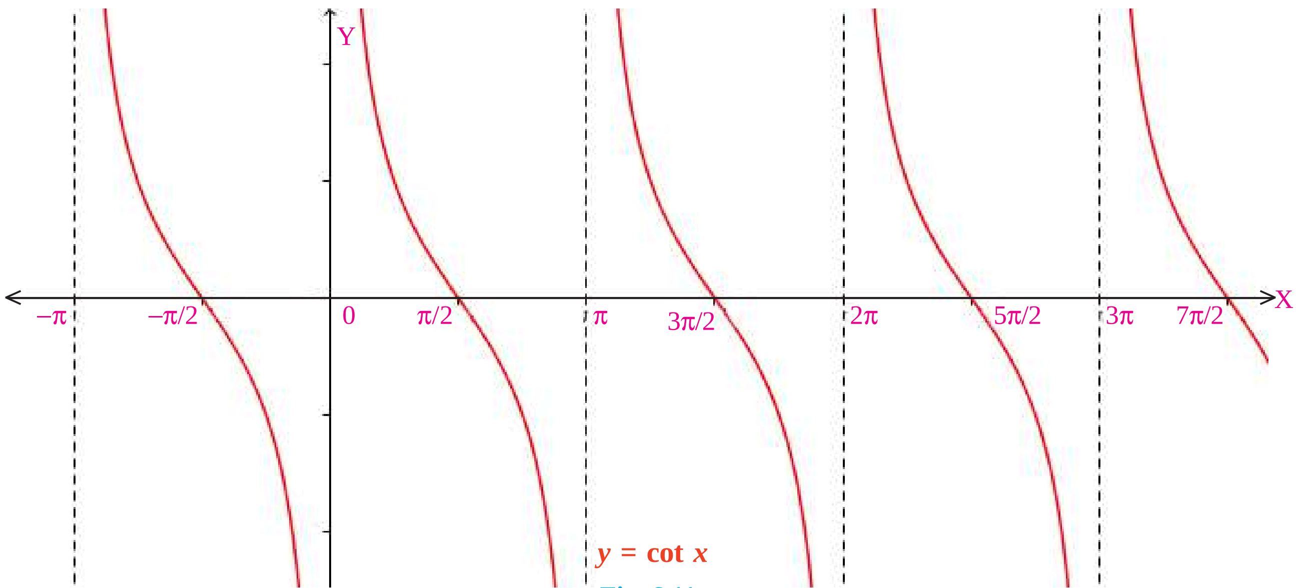


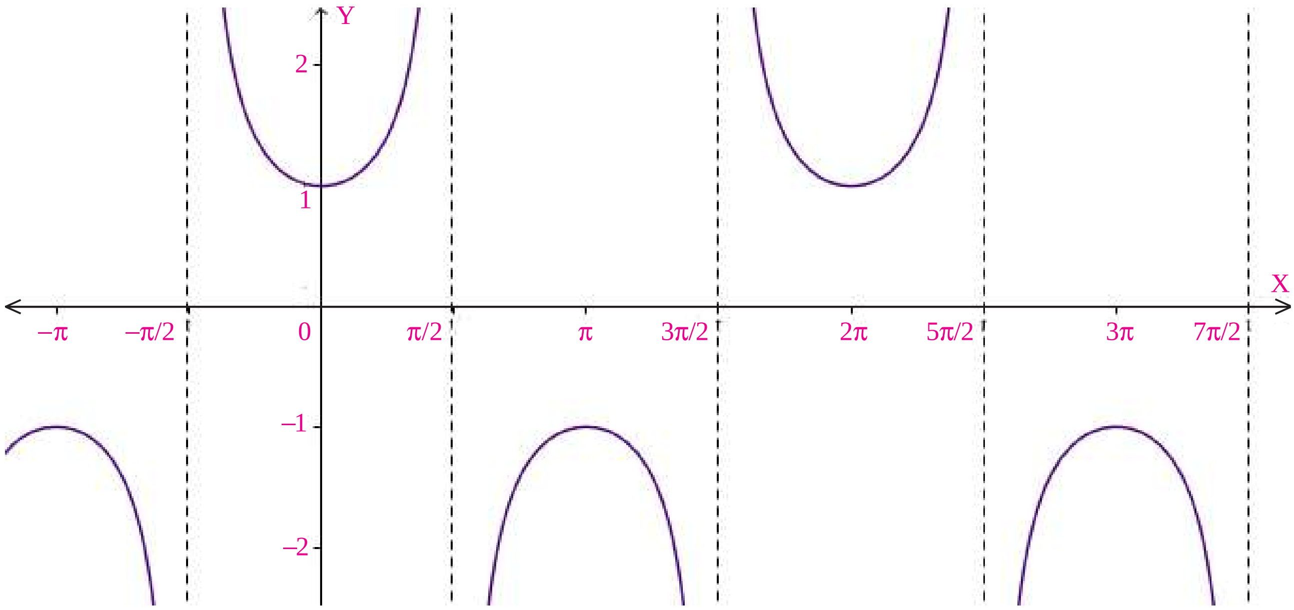
Fig. 6.11

5. Graph of $y = \sec x$

Like $y = \tan x$, this function is also not defined at $\frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ etc. Also note that $\sec x \to \infty$ as $x \to \frac{\pi}{2}$ in $\left(0, \frac{\pi}{2}\right)$ and $\sec x \to -\infty$ as $x \to \frac{\pi}{2}$ in $\left(\frac{\pi}{2}, \pi\right)$. Similarly for $\frac{3\pi}{2}, \frac{5\pi}{2}$ etc. We keep these points in view while drawing the graph of $y = \sec x$.

Table 6.8

x	$-\pi$	$-\pi/2$	0	$\pi/2$	π	$3\pi/2$	2π	$5\pi/2$	3π
$y = \sec x$	-1	not defined	1	not defined	-1	not defined	1	not defined	-1



$y = \sec x$

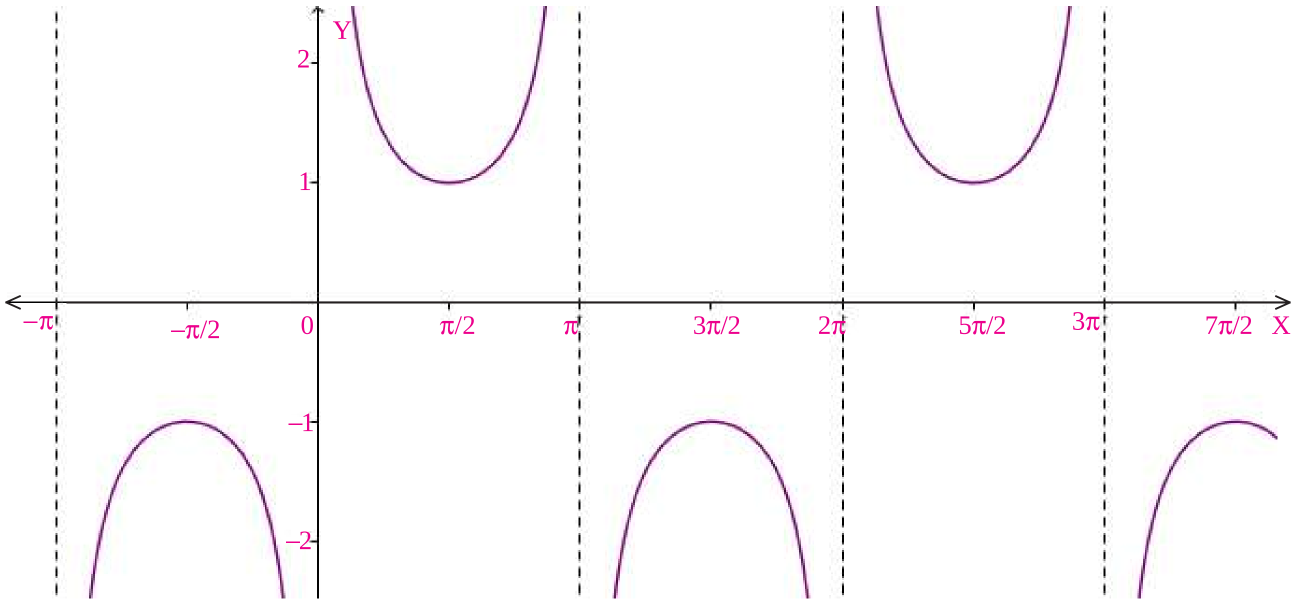
Fig. 6.12

6. Graph of $y = \text{cosec } x$

Like the function $y = \cot x$, this function is also not defined at all integral multiples of π like $-\pi$, 0, π, 2π etc. (since $\sin x = 0$ at these values of x). But note that $\text{cosec } x \to \infty$ as $x \to 0$ in $\left(0, \frac{\pi}{2}\right)$ and $\text{cosec } x \to -\infty$ as $x \to 0$ in $\left(\frac{-\pi}{2}, 0\right)$. Similarly for $\pi, 2\pi, 3\pi$ etc. We keep these things in mind while drawing the graph of $y = \text{cosec } x$.

Table 6.9

x	$-\pi$	$-\pi/2$	0	$\pi/2$	π	$3\pi/2$	2π	$5\pi/2$	3π
$y=\operatorname{cosec} x$	not defined	-1	not defined	1	not defined	-1	not defined	1	not defined



$y = \operatorname{cosec} x$

Fig. 6.13

Exercise 6(b)

I. Find the periods for the given 1 - 5 functions

1. $\cos(3x+5)+7$
2. $\tan 5x$
3. $\cos\left(\dfrac{4x+9}{5}\right)$
4. $|\sin x|$
5. $\tan(x + 4x + 9x + ... + n^2x)$ (n any positive integer)
6. Find a sine function whose period is $\dfrac{2}{3}$.
7. Find a cosine function whose period is 7.

II. Sketch the graph of the following functions

1. $\tan x$ between 0 and $\dfrac{\pi}{4}$
2. $\cos 2x$ in the interval $[0, \pi]$.

3. $\sin 2x$ in the interval $(0, \pi)$. **4.** $\sin x$ in the interval $[-\pi, +\pi]$

5. $\cos^2 x$ in $[0, \pi]$

III. Sketch the region enclosed by $y = \sin x$, $y = \cos x$ and X-axis in the interval $[0, \pi]$.

6.2 Trigonometric ratios of compound angles

In this section, we define a compound angle and give formulae to find the trigonometric ratios of compound angles.

6.2.1 Definition

*The algebraic sum of two or more angles is called a "**Compound angle**".*

If A, B, C are three angles then A + B, A – C, A + B + C, A – B + C etc. are compound angles.

6.2.2 Theorem : *If* A, B *are two real numbers, then*

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

Proof: We prove this theorem in various cases depending on the magnitudes of A, B, A + B.

Case (i): $A > 0$, $B > 0$ and $A + B < 2\pi$.

Consider a rectangular Cartesian system OXY. Let C be the circle in XY plane with centre at the origin O(0, 0) and radius 1 unit. Suppose the circle cuts OX at P. Then P = (1, 0). Let us take the points Q, R on this circle such that $\angle POQ = A$ and $\angle POR = A + B$ measured in anti-clockwise (positive) direction. Let S be the point on the circle such that $\angle POS = B$ measured in clockwise (negative) direction. Then the coordinates of Q, R, S are respectively (cos A, sin A), (cos (A + B), sin (A + B)), (cos (–B), sin (–B)).

Sub - Case (i): $A + B < \pi$.

Then the angles A, B, A + B are as shown in Fig. 6.14.

In triangle POR, OP = OR = 1 unit and $\angle POR = A + B$ and in triangle QOS, OQ = OS = 1 unit and $\angle QOS = A + B$. Therefore, the two triangles POR and QOS are congruent. Hence PR = QS .

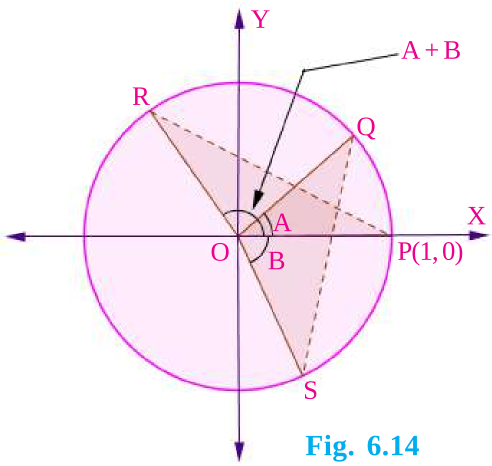


Fig. 6.14

Sub - Case (ii): $A + B = \pi$.

Then the points P, O, R are collinear (lie on X-axis) and the points Q, O, S are collinear as shown in Fig. 6.15.

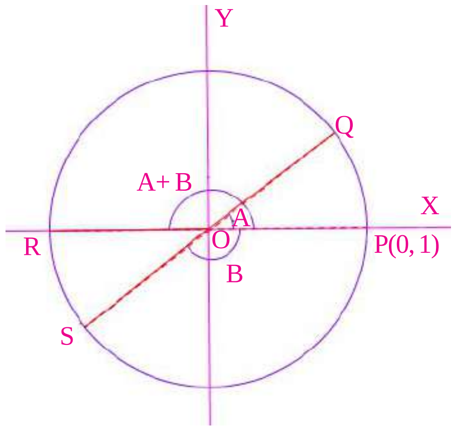


Fig. 6.15

Clearly PR = 2 = QS (diameters of the unit circle).

Sub - Case (iii): $A + B > \pi$ (Fig. 6.16).

Without loss of generality we can assume $A < \pi$.

In the triangle POR, $\angle POR = 2\pi - (A + B)$ (since $2\pi - (A + B) < \pi$) and

OP = OR = 1 unit and in the triangle QOS, $\angle QOS = 2\pi - (A + B)$ and OQ = OS = 1 unit.

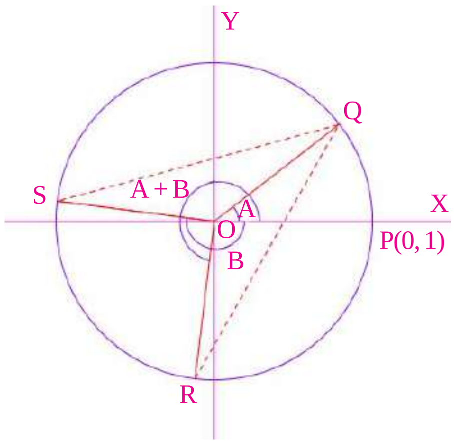


Fig. 6.16

Hence the two triangles are congruent. Therefore PR = QS.

In all the 3 sub-cases above, we get

$$PR = QS \Rightarrow PR^2 = QS^2.$$

$$\Rightarrow (\cos(A+B) - 1)^2 + (\sin(A+B) - 0)^2$$

$$= (\cos A - \cos(-B))^2 + (\sin A - \sin(-B))^2$$

$$\Rightarrow \cos^2(A+B) + 1 - 2\cos(A+B) + \sin^2(A+B)$$

$$= (\cos A - \cos B)^2 + (\sin A + \sin B)^2$$

$$\Rightarrow 2 - 2\cos(A+B) = \cos^2 A + \cos^2 B - 2\cos A\cos B + \sin^2 A + \sin^2 B + 2\sin A\sin B$$

$$= 2 - 2(\cos A\cos B - \sin A\sin B)$$

$$\Rightarrow \cos(A + B) = \cos A\cos B - \sin A\sin B. \qquad \ldots (1)$$

Case (ii): $A = 0$ (or $B = 0$) : We consider the case $A = 0$. The proof when $B = 0$ is similar.

Then $A + B = B$, $\cos A = 1$ and $\sin A = 0$. Thus,

$\text{Cos } A \text{ Cos } B - \sin A \sin B = \cos B = \cos(A + B)$.

Case (iii): $A \geq 0,\ B \geq 0$ and $A + B = 2\pi$

In this case, $\cos B = \cos(2\pi - A) = \cos A$ and

$\sin B = \sin(2\pi - A) = -\sin A$. Thus,

$$\cos A \cos B - \sin A \sin B = \cos^2 A + \sin^2 A = 1 = \cos 2\pi = \cos(A + B).$$

Case (iv): $A \in [0, \pi],\ B \in [\pi, 2\pi]$ (or $A \in [\pi, 2\pi),\ B \in [0, \pi]$).

Write $\theta = B - \pi$. Then $\theta \in [0, \pi]$. Hence

$$\cos(A + B) = \cos(A + \pi + \theta) = -\cos(A + \theta) = -\{\cos A \cos\theta - \sin A \sin\theta\}$$

(since $A + \theta \leq 2\pi$)

$$= \cos A \cos(\pi + \theta) - \sin A \sin(\pi + \theta)$$

$$= \cos A \cos B - \sin A \sin B.$$

Case (v): $A \in (\pi, 2\pi]$ and $B \in (\pi, 2\pi]$

Write $\alpha = A - \pi,\ \beta = B - \pi$. Then $0 \leq \alpha,\ \beta \leq \pi$ and hence

$$\cos(A + B) = \cos(\pi + \alpha + \pi + \beta) = \cos(2\pi + \alpha + \beta)$$

$$= \cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

$$= \cos(\pi + \alpha)\cos(\pi + \beta) - \sin(\pi + \alpha)\sin(\pi + \beta)$$

$$= \cos A \cos B - \sin A \sin B$$

Thus we have proved the theorem for all $A, B \in [0, 2\pi]$.

Finally, we prove the result in the general case.

Let $A, B \in \mathbf{R}$. Take $m = \left[\dfrac{A}{2\pi}\right]$ and $n = \left[\dfrac{B}{2\pi}\right]$. Then

$2\pi m \leq A < 2\pi(m + 1)$ and $2\pi n \leq B < 2\pi(n + 1)$.

Write $\alpha = A - 2\pi m$ and $\beta = B - 2n\pi$. Then $\alpha, \beta \in [0, 2\pi)$ and we have

$$\cos(A + B) = \cos(2\pi m + \alpha + 2n\pi + \beta) = \cos(\alpha + \beta)$$

$$= \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

Therefore $\cos(A + B) = \cos A \cos B - \sin A \sin B$.

The other formulae are derived from the above theorem as follows.

6.2.3 Corollary: For any $A, B \in \mathbf{R}$

(i) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(ii) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

(iii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

Proof: (i) $\cos(A - B) = \cos(A + (-B))$

$$= \cos A \cos(-B) - \sin A \sin(-B)$$

$$= \cos A \cos B + \sin A \sin B$$

since $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$

(ii) $\sin(A + B) = \cos\left(\frac{\pi}{2} - (A + B)\right) = \cos\left(\left(\frac{\pi}{2} - A\right) - B\right)$

$$= \cos\left(\frac{\pi}{2} - A\right)\cos B + \sin\left(\frac{\pi}{2} - A\right)\sin B$$

$$= \sin A \cos B + \cos A \sin B.$$

(iii) $\sin(A - B) = \sin(A + (-B))$

$$= \sin A \cos(-B) + \cos A \sin(-B)$$

$$= \sin A \cos B - \cos A \sin B.$$

6.2.4 Theorem

(i) *If none of* A, B *and* (A + B) *is an odd multiple of* $\frac{\pi}{2}$, *then*

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

(ii) *If none of* A, B *and* (A + B) *is an integral multiple of* π, *then*

$$\cot(A + B) = \frac{\cot B \cot A - 1}{\cot B + \cot A}$$

Proof

(i) Since none of A, B, A + B is an odd multiple of $\frac{\pi}{2}$, none of $\cos A$, $\cos B$, $\cos A + B$ is zero. Now

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}.$$

On dividing the numerator and the denominator in R.H.S. by cos A cos B, we get

$$\tan(A+B) = \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}} = \frac{\tan A + \tan B}{1 - \tan A . \tan B}.$$

(ii) Since none of A, B, (A + B) is an integral multiple of π, none of $\sin A, \sin B, \sin(A+B)$ is zero. Now

$$\cot(A+B) = \frac{\cos(A+B)}{\sin(A+B)} = \frac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}.$$

On dividing the numerator and denominator in R.H.S. by $\sin A \sin B$ we get

$$\cot(A+B) = \frac{\dfrac{\cos A \cos B}{\sin A \sin B} - \dfrac{\sin A \sin B}{\sin A \sin B}}{\dfrac{\sin A \cos B}{\sin A \sin B} + \dfrac{\cos A \sin B}{\sin A \sin B}} = \frac{\cot A \cot B - 1}{\cot B + \cot A}.$$

6.2.5 Note

1. If none of A, B, A + B is an odd multiple of $\frac{\pi}{2}$, then $\tan A$, $\tan B$ are defined and $\tan A \tan B \neq 1$ (since $\cos(A+B) \neq 0$) and hence the formula for $\tan(A+B)$ given in the above theorem is valid.
2. If none of A, B, A + B is an integral multiple of π, then $\cot A$, $\cot B$ are defined and $\cot B + \cot A \neq 0$ (since $\sin(A+B) \neq 0$) and hence the formula for $\cot(A+B)$ given in the above theorem is valid.

On replacing 'B' by '– B' in Theorem 6.2.4, we get the following.

6.2.6 Corollary

(i) If none of A, B, A – B is an odd multiple of $\frac{\pi}{2}$, then

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

(ii) If none of A, B, A – B is an integral multiple of π, then

$$\cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

6.2.7 Theorem: *For any two real numbers* A, B

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A.$

(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A.$

Proof (i) $\sin(A+B)\sin(A-B)$

$$= (\sin A \cos B + \cos A \sin B).(\sin A \cos B - \cos A \sin B)$$

$$= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B$$

$$= \sin^2 A(1 - \sin^2 B) - (1 - \sin^2 A)\sin^2 B$$

$$= \sin^2 A - \sin^2 B$$

$$= \left(1 - \cos^2 A\right) - \left(1 - \cos^2 B\right)$$
$$= \cos^2 B - \cos^2 A .$$

(ii) $\cos\left(A + B\right) \cos\left(A - B\right)$

$$= \left(\cos A \cos B - \sin A \sin B\right)\left(\cos A \cos B + \sin A \sin B\right)$$
$$= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$$
$$= \cos^2 A \left(1 - \sin^2 B\right) - \left(1 - \cos^2 A\right) \sin^2 B$$
$$= \cos^2 A - \sin^2 B$$
$$= \left(1 - \sin^2 A\right) - \left(1 - \cos^2 B\right)$$
$$= \cos^2 B - \sin^2 A .$$

Now we give the formulae for $\sin\left(A + B + C\right)$, $\cos\left(A + B + C\right)$,

$\tan\left(A + B + C\right)$ and $\cot\left(A + B + C\right)$ in the following.

6.2.8 Theorem: *If* A, B, C *are real numbers, then*

(i) $\sin\left(A + B + C\right) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C .$

(ii) $\cos\left(A + B + C\right) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C .$

(iii) *If none of* A, B, C *and* A + B + C *is an odd multiple of* $\frac{\pi}{2}$ *and at least one of* A + B, B + C, C + A *is not an odd multiple of* $\frac{\pi}{2}$, *then*

$$\tan\left(A + B + C\right) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}.$$

(iv) *If none of* A, B, C *and* A + B + C *is an integral multiple of* π, *then*

$$\cot\left(A + B + C\right) = \frac{\cot A + \cot B + \cot C - \cot A \cot B \cot C}{1 - \cot A \cot B - \cot B \cot C - \cot C \cot A}.$$

Proof

(i) $\sin\left(A + B + C\right) = \sin\left(\left(A + B\right) + C\right)$

$$= \sin\left(A + B\right)\cos C + \cos\left(A + B\right) \sin C$$
$$= \left(\sin A \cos B + \cos A \sin B\right) \cos C + \left(\cos A \cos B - \sin A \sin B\right) \sin C.$$
$$= \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$$

This formula can be written as,

$$\sin(A + B + C) = \Sigma(\sin A \cos B \cos C) - \sin A \sin B \sin C$$

(ii) $\cos(A + B + C) = \cos((A + B) + C)$

$$= \cos(A + B)\cos C - \sin(A + B)\sin C$$

$$= (\cos A \cos B - \sin A \sin B)\cos C - (\sin A \cos B + \cos A \sin B)\sin C$$

$$= \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C$$

This can be written as

$$\cos(A + B + C) = \cos A \cos B \cos C - \Sigma \cos A \sin B \sin C$$

(iii) Suppose none of A, B, C, $A + B + C$ is an odd multiple of $\frac{\pi}{2}$ and assume, without loss of generality, that $A + B$ is not an odd multiple of $\frac{\pi}{2}$. Then

$$\tan(A + B + C) = \tan((A + B) + C) = \frac{\tan(A + B) + \tan C}{1 - \tan(A + B)\tan C}$$

$$= \frac{\dfrac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \left(\dfrac{\tan A + \tan B}{1 - \tan A \tan B}\right)\tan C}$$

$$= \frac{\tan A + \tan B + \tan C(1 - \tan A \tan B)}{1 - \tan A \tan B - (\tan A + \tan B)\tan C}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}.$$

This can be written as

$$\tan(A + B + C) = \frac{\Sigma \tan A - \Pi \tan A}{1 - \Sigma \tan A \tan B} = \frac{s_1 - s_3}{1 - s_2}$$

In the above formula,

s_1 = Sum of the tangents taken **one** at a time.

s_2 = Sum of the products of the tangents taken **two** at a time.

s_3 = Sum of the products of the tangents taken **three** at a time.

(iv) Assume that none of A, B, C, $A + B + C$ is an integral multiple of π and assume, without loss of generality, that $A + B$ is not a multiple of π, then

$$\cot(A + B + C) = \cot((A + B) + C) = \frac{\cot(A + B)\cot C - 1}{\cot C + \cot(A + B)}$$

$$= \frac{\left(\frac{\cot A \cot B - 1}{\cot B + \cot A}\right)\cot C - 1}{\cot C + \frac{\cot A \cot B - 1}{\cot B + \cot A}}$$

$$= \frac{\cot A \cot B \cot C - \cot C - \cot B - \cot A}{\cot C \cot B + \cot C \cot A + \cot A \cot B - 1}$$

$$= \frac{\cot A + \cot B + \cot C - \cot A \cot B \cot C}{1 - \cot A \cot B - \cot B \cot C - \cot C \cot A}.$$

This can be written as

$$\cot(A + B + C) = \frac{\Sigma \cot A - \Pi \cot A}{1 - \Sigma \cot A \cot B} = \frac{s_1 - s_3}{1 - s_2}.$$

In the above formula

s_1 = Sum of the cotangents taken **one** at a time

s_2 = Sum of the products of the cotangents taken **two** at a time

s_3 = Sum of the products of the cotangents taken **three** at a time

6.2.9 Solved Problems

1. Problem: *Find the values of* $\sin 75^0$, $\cos 75^0$, $\tan 75^0$ *and* $\cot 75^0$.

Solution

(i) $\sin 75^0 = \sin(45^0 + 30^0) = \sin 45^0 \cos 30^0 + \cos 45^0 \sin 30^0$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

(ii) $\cos 75^0 = \cos(45^0 + 30^0) = \cos 45^0 \cos 30^0 - \sin 45^0 \sin 30^0$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

(iii) $\tan 75^0 = \frac{\sin 75^0}{\cos 75^0} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)} = 2 + \sqrt{3}.$

(iv) $\cot 75^0 = \frac{1}{\tan 75^0} = \frac{1}{2+\sqrt{3}} = 2 - \sqrt{3}.$

2. Problem: *If* $0 < A, B < 90^0$. $\cos A = \frac{5}{13}$ *and* $\sin B = \frac{4}{5}$ *then find* $\sin(A + B)$.

Solution: $0 < A < 90^0$ and $\cos A = \frac{5}{13} \Rightarrow \sin A = \frac{12}{13}.$

$0 < B < 90^0$ and $\sin B = \frac{4}{5} \Rightarrow \cos B = \frac{3}{5}.$

$\therefore \sin(A+B) = \sin A \cos B + \cos A \sin B$

$$= \frac{12}{13} \cdot \frac{3}{5} + \frac{5}{13} \cdot \frac{4}{5} = \frac{56}{65}.$$

3. Problem: *Prove that* $\sin^2\left(52\frac{1}{2}\right)^0 - \sin^2\left(22\frac{1}{2}\right)^0 = \frac{\sqrt{3}+1}{4\sqrt{2}}$.

Solution: Put $A = \left(52\frac{1}{2}\right)^0$ and $B = \left(22\frac{1}{2}\right)^0$. Then

$\sin^2\left(52\frac{1}{2}\right)^0 - \sin^2\left(22\frac{1}{2}\right)^0$

$= \sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$ (from Theorem 6.2.7(i))

$= \sin 75^0 \sin 30^0 = \frac{\sqrt{3}+1}{2\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}+1}{4\sqrt{2}}.$

4. Problem: *Prove that* $\tan 70^0 - \tan 20^0 = 2\tan 50^0$.

Solution: $\tan 50^0 = \tan\left(70^0 - 20^0\right) = \frac{\tan 70^0 - \tan 20^0}{1+\tan 70^0 \,.\, \tan 20^0}$

$\Rightarrow \tan 70^0 - \tan 20^0 = \tan 50^0\left(1+\tan 70^0 \,.\, \tan\left(90^0 - 70^0\right)\right)$

$= \tan 50^0\left(1+\tan 70^0 \,.\, \cot 70^0\right)$

$= 2\tan 50^0.$

5. Problem: *If* $A + B = \frac{\pi}{4}$, *then prove that*

(i) $(1+\tan A)(1+\tan B) = 2$, (ii) $(\cot A - 1)(\cot B - 1) = 2$.

Solution

(i) $A + B = \frac{\pi}{4}$

$\Rightarrow \tan(A+B) = \tan\frac{\pi}{4} = 1$

$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1 \Rightarrow \tan A + \tan B = 1 - \tan A \tan B$

$\Rightarrow \tan A + \tan B + \tan A \tan B = 1$ ... (1)

Now, $(1+\tan A)(1+\tan B) = 1 + \tan A + \tan B + \tan A \tan B = 2$ (from (1))

(ii) $A + B = \frac{\pi}{4} \Rightarrow \cot(A+B) = \cot\frac{\pi}{4} = 1$

$\Rightarrow \frac{\cot A \cot B - 1}{\cot B + \cot A} = 1 \Rightarrow \cot A \cot B - 1 = \cot A + \cot B$

$\Rightarrow \cot A \cot B - \cot A - \cot B = 1$ ... (2)

Now, $(\cot A - 1)(\cot B - 1) = \cot A \cot B - \cot A - \cot B + 1 = 2$ (from (2)).

6. Problem: *If* $\sin\alpha = \frac{1}{\sqrt{10}}, \sin\beta = \frac{1}{\sqrt{5}}$ *and α, β are acute, show that α + β = π/4.*

Solution

Given α is acute and $\sin\alpha = \frac{1}{\sqrt{10}} \Rightarrow \tan\alpha = \frac{1}{3}$.

β is acute and $\sin\beta = \frac{1}{\sqrt{5}} \Rightarrow \tan\beta = \frac{1}{2}$.

$$\text{Therefore } \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\cdot\frac{1}{2}} = 1$$

$\Rightarrow \quad \alpha + \beta = \pi/4.$

7. Problem: *If* $\sin A = \frac{12}{13}, \ \cos B = \frac{3}{5}$ *and neither* A *nor* B *is in the first quadrant, then find the quadrant in which* A + B *lies.*

Solution: From hypothesis, A lies in the second quadrant and B in the fourth quadrant

so that $2n\pi + \frac{\pi}{2} < A < 2n\pi + \pi$

and $2m\pi + \frac{3\pi}{2} < B < (2m+2)\pi$ for some integers *m, n*.

On adding, we get

$$2n\pi + 2m\pi + 2\pi < A + B < 2n\pi + 2m\pi + 2\pi + \pi$$

That is $2k\pi < A + B < 2k\pi + \pi$ where $k = m + n + 1$

Therefore, A + B lies either in first or in second quadrant. ... (1)

Now $\cos^2 A + \sin^2 A = 1 \Rightarrow \cos^2 A + \frac{144}{169} = 1 \Rightarrow \cos^2 A = 1 - \frac{144}{169} = \frac{25}{169}$

$$\Rightarrow \cos A = \pm\frac{5}{13}$$

$$\Rightarrow \cos A = -\frac{5}{13} \text{ (Since A lies in 2nd quadrant)}$$

Similarly, $\cos^2 B + \sin^2 B = 1 \Rightarrow \sin^2 B = 1 - \frac{9}{25} = \frac{16}{25}$

$$\Rightarrow \sin B = \pm\frac{4}{5} \Rightarrow \sin B = -\frac{4}{5}$$

(Since B lies in 4th quadrant)

Now, $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$= \left(\frac{-5}{13}\right)\left(\frac{3}{5}\right) - \left(\frac{12}{13}\right)\left(\frac{-4}{5}\right) = \frac{33}{65}.$$

Therefore, from (1), $A+B$ lies in the first quadrant.

8. Problem: *Find* (i) $\tan\left(\frac{\pi}{4}+A\right)$ interms of $\tan A$ and

(ii) $\cot\left(\frac{\pi}{4}+A\right)$ interms of $\cot A$.

Solution

(i) $$\tan\left(\frac{\pi}{4}+A\right) = \frac{\tan\frac{\pi}{4}+\tan A}{1-\tan\frac{\pi}{4}\tan A} = \frac{1+\tan A}{1-\tan A} \quad (\text{provided } \tan A \neq 1)$$

(ii) $$\cot\left(\frac{\pi}{4}+A\right) = \frac{\cot\frac{\pi}{4}\cot A - 1}{\cot A + \cot\frac{\pi}{4}} = \frac{\cot A - 1}{\cot A + 1} \quad (\text{only when } \cot A + 1 \neq 0).$$

9. Problem: *Prove that* $\frac{\cos 9^0 + \sin 9^0}{\cos 9^0 - \sin 9^0} = \cot 36^0$.

Solution: L.H.S. $= \frac{\cos 9^0 + \sin 9^0}{\cos 9^0 - \sin 9^0}$

$= \frac{1+\tan 9^0}{1-\tan 9^0}$ (on dividing numerator and denominator by $\cos 9^0$)

$= \tan(45^0 + 9^0)$ (by problem 8(i))

$= \tan 54^0 = \tan(90-36^0) = \cot 36^0 =$ R.H.S.

10. Problem: *Show that* $\cos 42^0 + \cos 78^0 + \cos 162^0 = 0$.

Solution: L.H.S. $= \cos(60^0 - 18^0) + \cos(60^0 + 18^0) + \cos(180^0 - 18^0)$

$= \cos 60^0 \cos 18^0 + \sin 60^0 \sin 18^0 + \cos 60^0 \cos 18^0 - \sin 60^0 \sin 18^0 - \cos 18^0$

$= 2\cos 60^0 \cos 18^0 - \cos 18^0 = 2.\frac{1}{2}.\cos 18^0 - \cos 18^0 = 0$.

11. Problem: *Express* $\sqrt{3}\sin\theta + \cos\theta$ *as a sine of an angle.*

Solution: $\sqrt{3}\sin\theta + \cos\theta = 2\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right)$

$$= 2(\cos\frac{\pi}{6}\sin\theta + \sin\frac{\pi}{6}\cos\theta)$$

$$= 2 . \sin(\theta + \frac{\pi}{6}).$$

12. Problem: *Prove that* $\sin^2\theta + \sin^2(\theta + \frac{\pi}{3}) + \sin^2(\theta - \frac{\pi}{3}) = \frac{3}{2}$.

Solution: L.H.S. $= \sin^2\theta + \sin^2(\theta + \frac{\pi}{3}) + \sin^2(\theta - \frac{\pi}{3})$

$$= \sin^2\theta + (\sin\theta\cos\frac{\pi}{3} + \cos\theta\sin\frac{\pi}{3})^2 + (\sin\theta\cos\frac{\pi}{3} - \cos\frac{\pi}{3}\sin\theta)^2$$

$$= \sin^2\theta + 2(\sin^2\theta\cos^2\frac{\pi}{3} + \cos^2\theta\sin^2\frac{\pi}{3})$$

$$= \sin^2\theta + 2\left(\sin^2\theta.\frac{1}{4} + \cos^2\theta.\frac{3}{4}\right)$$

$$= \sin^2\theta + \frac{1}{2}\sin^2\theta + \frac{3}{2}\cos^2\theta$$

$$= \frac{3}{2}\sin^2\theta + \frac{3}{2}\cos^2\theta = \frac{3}{2}(\sin^2\theta + \cos^2\theta)$$

$$= \frac{3}{2} = \text{R.H.S.}$$

13. Problem: *If* A, B, C *are the angles of a triangle and if none of them is equal to* $\frac{\pi}{2}$, *then prove that*

(i) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

(ii) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$.

Solution

(i) Given $A + B + C = \pi$

$$\Rightarrow A + B = \pi - C \Rightarrow \tan(A + B) = \tan(\pi - C)$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

$$\Rightarrow \tan A + \tan B = -\tan C(1 - \tan A \tan B) = -\tan C + \tan A \tan B \tan C$$

$$\Rightarrow \tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

(ii) Replacing $\tan A$ by $\frac{1}{\cot A}$ etc. in (i) above, we get

$$\frac{1}{\cot A} + \frac{1}{\cot B} + \frac{1}{\cot C} = \frac{1}{\cot A . \cot B \cot C}$$

$$\Rightarrow \cot A \cot B + \cot B \cot C + \cot C \cot A = 1.$$

14. Problem: *Let* ABC *be a triangle such that* $\cot A + \cot B + \cot C = \sqrt{3}$. *Then prove that* ABC *is an equilateral triangle.*

Solution: Given that $A + B + C = 180^0$, by problem 13(ii),

we get $\sum \cot A \cot B = 1$. Now

$$\sum(\cot A - \cot B)^2 = \sum(\cot^2 A + \cot^2 B - 2\cot A \cot B)$$

$$= 2\cot^2 A + 2\cot^2 B + 2\cot^2 C - 2\cot A\cot B - 2\cot B\cot C - 2\cot C\cot A$$

(on expanding)

$$= 2\left\{(\cot A + \cot B + \cot C)^2 - 2\cot A\cot B - 2\cot B\cot C - 2\cot C\cot A\right\}$$
$$- 2(\cot A\cot B + \cot B\cot C + \cot C\cot A)$$

$$= 2(\cot A + \cot B + \cot C)^2 - 6(\cot A\cot B + \cot B\cot C + \cot C\cot A)$$

$$= 2\cdot 3 - 6.1 = 0$$

$\Rightarrow \cot A = \cot B = \cot C$

$\Rightarrow \cot A = \cot B = \cot C = \dfrac{\sqrt{3}}{3} = \dfrac{1}{\sqrt{3}}$ (since $\cot A + \cot B + \cot C = \sqrt{3}$)

$\Rightarrow A = B = C = \dfrac{\pi}{3}$ (since each angle lies in the interval $[0, \pi]$).

15. Problem: *Suppose* $x = \tan A$, $y = \tan B$, $z = \tan C$.

Suppose none of A, B, C, A – B, B – C, C – A *is an odd multiple of* $\dfrac{\pi}{2}$.

Then prove that $\sum\left(\dfrac{x-y}{1+xy}\right) = \Pi\left(\dfrac{x-y}{1+xy}\right)$

Solution: Observe that $\dfrac{x-y}{1+xy} = \dfrac{\tan A - \tan B}{1+\tan A\tan B} = \tan(A-B)$ etc. ... (1)

Write $P = A - B$, $Q = B - C$, $R = C - A$. Then $P + Q + R = 0$.

$\Rightarrow P + Q = -R \Rightarrow \tan(P+Q) = \tan(-R)$

$\Rightarrow \dfrac{\tan P + \tan Q}{1 - \tan P.\tan Q} = -\tan R$

$\Rightarrow \tan P + \tan Q = -\tan R(1 - \tan P\tan Q)$

$= -\tan R + \tan P\tan Q\tan R$

$\Rightarrow \tan P + \tan Q + \tan R = \tan P\tan Q\tan R$

$\Rightarrow \sum \tan P = \Pi \tan P$

$\Rightarrow \sum \tan(A-B) = \Pi \tan(A-B)$

$\Rightarrow \sum\left(\dfrac{x-y}{1+xy}\right) = \Pi\left(\dfrac{x-y}{1+xy}\right)$ from (1).

6.2.10 Note

1. In problem 1, we obtained the trigonometric ratios of 75^0.

Since $15^0 = 90^0 - 75^0$ and $105^0 = 180^0 - 75^0$, we also get that

(i) $\sin 15^0 = \cos 75^0 = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

(ii) $\cos 15^0 = \sin 75^0 = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

(iii) $\tan 15^0 = \cot 75^0 = \dfrac{1}{\tan 75^0} = \dfrac{1}{2+\sqrt{3}} = 2-\sqrt{3}$.

(iv) $\cot 15^0 = \tan 75^0 = 2+\sqrt{3}$.

(v) $\sin 105^0 = \sin\left(180^0 - 75^0\right) = \sin 75^0 = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

(vi) $\cos 105^0 = -\cos 75^0 = -\dfrac{\left(\sqrt{3}-1\right)}{2\sqrt{2}}$.

(vii) $\tan 105^0 = -\tan 75^0 = -\left(2+\sqrt{3}\right)$.

(viii) $\cot 105^0 = -\cot 75^0 = -\left(2-\sqrt{3}\right)$.

6.2.11 Extreme values of trigonometric functions

We have observed that, for any $\theta \in \mathbf{R}$, $-1 \le \sin\theta \le 1$.

Also we know that $\sin\dfrac{\pi}{2} = 1$ and $\sin\left(-\dfrac{\pi}{2}\right) = -1$. Hence the maximum and minimum values of $\sin\theta$ are respectively 1 and -1 as θ ranges over $\mathbf{R}$. Each of them is called an extreme value of $\sin\theta$. Similarly, the maximum and minimum values of $\cos\theta$ are respectively 1 and -1 over $\mathbf{R}$.

6.2.12 Theorem

If $a, b, c \in \mathbf{R}$ such that $a^2 + b^2 \ne 0$, then the maximum and minimum values of $a\sin x + b\cos x + c$ are respectively $c + \sqrt{a^2+b^2}$ and $c - \sqrt{a^2+b^2}$ over $\mathbf{R}$.

Solution: Define $f(x) = a\sin x + b\cos x + c$ for all $x \in \mathbf{R}$.

Put $a = r\cos\theta$ and $b = r\sin\theta$ where $r = \sqrt{a^2+b^2}$. Then

$$f(x) = r\cos\theta\sin x + r\sin\theta\cos x + c$$
$$= r\left(\cos\theta\sin x + \sin\theta\cos x\right) + c$$
$$= r\sin(\theta + x) + c \qquad \text{... (1)}$$

We know that $-1 \le \sin(\theta + x) \le 1$. So that

$$-r \le r\sin(\theta + x) \le r$$

and hence $c - r \le \{c + r\sin(\theta + x)\} \le c + r$ from (1).

Hence the maximum and minimum values of f over $\mathbf{R}$ are respectively $c + \sqrt{a^2 + b^2}$ and $c - \sqrt{a^2 + b^2}$.

6.2.13 Note

From the above theorem, we get that the range of the function f is $\left[c - \sqrt{a^2 + b^2},\ c + \sqrt{a^2 + b^2}\right]$ (since f is "continuous" on $\mathbf{R}$)

6.2.14 Example

Find the maximum and minimum values of

(i) $3\sin x - 4\cos x$ (ii) $\cos\left(x + \frac{\pi}{3}\right) + 2\sqrt{2}\sin\left(x + \frac{\pi}{3}\right) - 3$

Solution

(i) From Theorem 6.2.12, we get that the maximum value of $3\sin x - 4\cos x$ is $\sqrt{9 + 16} = 5$ and the minimum value is $-\sqrt{9 + 16} = -5$.

(ii) Again, from Theorem 6.2.12, we get that the maximum value of $\cos\alpha + 2\sqrt{2}\sin\alpha - 3$ (where $\alpha = x + \frac{\pi}{3}$) is $-3 + \sqrt{1 + 8} = 0$ and the minimum value is $-3 - \sqrt{1 + 8} = -6$.

Exercise 6(c)

I.1. Simplify the following

(i) $\cos 100^0 \cos 40^0 + \sin 100^0 . \sin 40^0$ (ii) $\frac{\cot 55^0 \cot 35^0 - 1}{\cot 55^0 + \cot 35^0}$

(iii) $\tan\left(\frac{\pi}{4} + \theta\right) . \tan\left(\frac{\pi}{4} - \theta\right)$ (iv) $\tan 75^0 + \cot 75^0$

(v) $\sin 1140^0 \cos 390^0 - \cos 780^0 \sin 750^0$

2. Express

(i) $\frac{\left(\sqrt{3}\cos 25^0 + \sin 25^0\right)}{2}$ as a sine of an angle.

(ii) $(\cos\theta - \sin\theta)$ as a cosine of an angle.

(iii) $\tan\theta$ in terms of $\tan\alpha$, if $\sin(\theta+\alpha) = \cos(\theta+\alpha)$.

3. (i) If $\tan\theta = \dfrac{\cos 11^0 + \sin 11^0}{\cos 11^0 - \sin 11^0}$ and θ is in the third quadrant, find θ.

(ii) If $0^0 < A, B < 90^0$, such that $\cos A = \dfrac{5}{13}$ and $\sin B = \dfrac{4}{5}$, find the value of $\sin(A - B)$.

(iii) What is the value of $\tan 20^0 + \tan 40^0 + \sqrt{3}\tan 20^0 \tan 40^0$?

(iv) Find the value of $\tan 56^0 - \tan 11^0 - \tan 56^0 \tan 11^0$.

(v) Evaluate $\sum \dfrac{\sin(A+B)\sin(A-B)}{\cos^2 A \cos^2 B}$; if none of $\cos A$, $\cos B$, $\cos C$ is zero.

(vi) Evaluate $\sum \dfrac{\sin(C-A)}{\sin C \sin A}$ if none of $\sin A$, $\sin B$, $\sin C$ is zero.

4. Prove that

(i) $\cos 35^0 + \cos 85^0 + \cos 155^0 = 0$

(ii) $\tan 72^0 = \tan 18^0 + 2\tan 54^0$

(iii) $\sin 750^0 \cos 480^0 + \cos 120^0 \cos 60^0 = \dfrac{-1}{2}$

(iv) $\cos A + \cos\left(\dfrac{4\pi}{3} - A\right) + \cos\left(\dfrac{4\pi}{3} + A\right) = 0$

(v) $\cos^2\theta + \cos^2\left(\dfrac{2\pi}{3} + \theta\right) + \cos^2\left(\dfrac{2\pi}{3} - \theta\right) = \dfrac{3}{2}$

5. Evaluate

(i) $\sin^2 82\dfrac{1}{2}^0 - \sin^2 22\dfrac{1}{2}^0$ (ii) $\cos^2 112\dfrac{1}{2}^0 - \sin^2 52\dfrac{1}{2}^0$

(iii) $\sin^2\left(\dfrac{\pi}{8} + \dfrac{A}{2}\right) - \sin^2\left(\dfrac{\pi}{8} - \dfrac{A}{2}\right)$ (iv) $\cos^2 52\dfrac{1}{2}^0 - \sin^2 22\dfrac{1}{2}^0$.

6. Find the minimum and maximum values of

(i) $3\cos x + 4\sin x$ (ii) $\sin 2x - \cos 2x$

7. Find the range of

(i) $7\cos x - 24\sin x + 5$ (ii) $13\cos x + 3\sqrt{3}\sin x - 4$

II.

1. (i) If $\cos\alpha = \frac{-3}{5}$ and $\sin\beta = \frac{7}{25}$, where $\frac{\pi}{2} < \alpha < \pi$ and $0 < \beta < \frac{\pi}{2}$, then find the values of $\tan(\alpha+\beta)$ and $\sin(\alpha+\beta)$.

(ii) If $0 < A < B < \frac{\pi}{4}$ and $\sin(A+B) = \frac{24}{25}$ and $\cos(A-B) = \frac{4}{5}$, then find the value of $\tan 2A$.

(iii) If $A+B$, A are acute angles such that $\sin(A+B) = \frac{24}{25}$ and $\tan A = \frac{3}{4}$, then find the value of $\cos B$.

(iv) If $\tan\alpha - \tan\beta = m$ and $\cot\alpha - \cot\beta = n$, then prove that $\cot(\alpha-\beta) = \frac{1}{m} - \frac{1}{n}$.

(v) If $\tan(\alpha-\beta) = \frac{7}{24}$ and $\tan\alpha = \frac{4}{3}$, where α and β are in the first quadrant prove that $\alpha + \beta = \pi/2$.

2. (i) Find the expansion of $\sin(A+B-C)$.

(ii) Find the expansion of $\cos(A-B-C)$.

(iii) In a ΔABC, A is obtuse. If $\sin A = \frac{3}{5}$ and $\sin B = \frac{5}{13}$, then show that $\sin C = \frac{16}{65}$.

(iv) If $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \frac{a+b}{a-b}$, then prove that $a\tan\beta = b\tan\alpha$.

III.

1. (i) If $A - B = \frac{3\pi}{4}$, then show that $(1-\tan A)(1+\tan B) = 2$.

(ii) If $A+B+C = \frac{\pi}{2}$ and if none of A, B, C is an odd multiple of $\frac{\pi}{2}$, then prove that

(a) $\cot A + \cot B + \cot C = \cot A \cot B \cot C$

(b) $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$ and hence show that

(c) $\sum \frac{\cos(B+C)}{\cos B \cos C} = 2$.

2. (i) Prove that $\sin^2\alpha + \cos^2(\alpha+\beta) + 2\sin\alpha\sin\beta\cos(\alpha+\beta)$ is independent of α.

(ii) Prove that $\cos^2(\alpha-\beta) + \cos^2\beta - 2\cos(\alpha-\beta)\cos\alpha\cos\beta$ is independent of β.

6.3 Trigonometric ratios of multiple and sub-multiple angles

In this section, we derive formulae for the trigonometric ratios of multiple angles 2A, 3A, ... in terms of those of A. Also we discuss about the trigonometric ratios of the sub-multiple angles $\frac{A}{2}, \frac{A}{3}, ...$ of A.

6.3.1 Definition

If A is an angle, then its integral multiples 2A, 3A, 4A, ... are called "Multiple angles of A" and the multiples of A by fractions like $\frac{1}{2}, \frac{1}{3}, ...$ are called "submultiple" angles of A.

6.3.2 Theorem

Let A be any real number. Then

(i) $\sin 2A = 2\sin A\cos A$

(ii) $\cos 2A = \cos^2 A - \sin^2 A$
$= 2\cos^2 A - 1$
$= 1 - 2\sin^2 A$

(iii) *If A and 2A are not odd multiples of $\frac{\pi}{2}$, then*

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

(iv) *If 2A is not an integral multiple of π, then*

$$\cot 2A = \frac{\cot^2 A - 1}{2\cot A}$$

Proof: (i) We know that $\sin(A+B) = \sin A\cos B + \cos A\sin B$.

Hence, $\sin 2A = \sin(A+A) = \sin A\cos A + \cos A\sin A$
$= 2\sin A\cos A$.

(ii) Similarly, $\cos 2A = \cos(A+A) = \cos A.\cos A - \sin A.\sin A$
$= \cos^2 A - \sin^2 A$.
$= \cos^2 A - (1-\cos^2 A) = 2\cos^2 A - 1$
$= 2(1-\sin^2 A) - 1$
$= 1 - 2\sin^2 A$.

(iii) $\tan 2A = \tan(A + A) = \dfrac{\tan A + \tan A}{1 - \tan A \,.\, \tan A}$

$$= \frac{2\tan A}{1 - \tan^2 A}.$$

(iv) $\cot 2A = \cot(A + A) = \dfrac{\cot A \,.\, \cot A - 1}{\cot A + \cot A} = \dfrac{\cot^2 A - 1}{2\cot A}.$

6.3.3 Theorem

For any real number A, *which is not an odd multiple of* $\dfrac{\pi}{2}$,

(i) $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$ (ii) $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

Proof

(i) From Theorem 6.3.2, $\sin 2A = 2\sin A\cos A$

$$= \frac{2\sin A\cos A}{\cos^2 A + \sin^2 A}$$

$$= \frac{\dfrac{2\sin A\cos A}{\cos^2 A}}{\dfrac{\cos^2 A + \sin^2 A}{\cos^2 A}} = \frac{2\tan A}{1 + \tan^2 A}.$$

(ii) Also, $\cos 2A = \cos^2 A - \sin^2 A$

$$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \frac{\dfrac{\cos^2 A - \sin^2 A}{\cos^2 A}}{\dfrac{\cos^2 A + \sin^2 A}{\cos^2 A}} = \frac{1 - \tan^2 A}{1 + \tan^2 A}.$$

On replacing A by $\dfrac{A}{2}$ in the above Theorems 6.3.2, 6.3.3, we get

6.3.4 Corollary

If $\dfrac{A}{2}$ *is not an odd multiple of* $\dfrac{\pi}{2}$, *then*

(i) $\sin A = 2\sin\dfrac{A}{2}\cos\dfrac{A}{2} = \dfrac{2\tan\dfrac{A}{2}}{1 + \tan^2\dfrac{A}{2}}.$

(ii) $\cos A = \cos^2\dfrac{A}{2} - \sin^2\dfrac{A}{2} = 2\cos^2\dfrac{A}{2} - 1 = 1 - 2\sin^2\dfrac{A}{2}.$

$$= \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}.$$

(iii) If $\frac{A}{2}$ *and* A *are not odd multiples of* $\frac{\pi}{2}$, *then* $\tan A = \frac{2\tan A/2}{1-\tan^2 A/2}$.

(iv) *If* A *is not an integral multiple of* π, *then* $\cot A = \frac{\cot^2 A/2 - 1}{2\cot A/2}$.

Now, we derive formulae for $\sin 3A$, $\cos 3A$, $\tan 3A$ and $\cot 3A$ in the following.

6.3.5 Theorem

For any real number A,

(i) $\sin 3A = 3\sin A - 4\sin^3 A$

(ii) $\cos 3A = 4\cos^3 A - 3\cos A$

(iii) *If* 3A *is not an odd multiple of* $\frac{\pi}{2}$, *then* $\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

(iv) *If* 3A *is not an integral multiple of* π, *then* $\cot 3A = \frac{3\cot A - \cot^3 A}{1-3\cot^2 A}$

Proof : (i)

$$\sin 3A = \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A$$
$$= 2\sin A \cos^2 A + (1 - 2\sin^2 A)\sin A$$
$$= 2\sin A(1-\sin^2 A) + \sin A - 2\sin^3 A$$
$$= 3\sin A - 4\sin^3 A.$$

(ii)

$$\cos 3A = \cos(2A + A) = \cos 2A \cos A - \sin 2A \sin A$$
$$= (2\cos^2 A - 1)\cos A - 2\sin^2 A \cos A$$
$$= 2\cos^3 A - \cos A - 2\cos A(1-\cos^2 A)$$
$$= 4\cos^3 A - 3\cos A.$$

(iii) Assume that A is not an odd multiple of $\frac{\pi}{2}$. Then

$$\tan 3A = \tan(2A + A) = \frac{\tan 2A + \tan A}{1 - \tan 2A . \tan A}$$

$$= \frac{\frac{2\tan A}{1-\tan^2 A} + \tan A}{1 - \frac{2\tan A}{1-\tan^2 A}.\tan A} = \frac{2\tan A + \tan A(1-\tan^2 A)}{1-\tan^2 A - 2\tan^2 A}$$

$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

(iv) $\cot 3A = \cot(2A + A) = \dfrac{\cot 2A \cot A - 1}{\cot A + \cot 2A}$

$$= \frac{\left(\dfrac{\cot^2 A - 1}{2\cot A}\right)\cot A - 1}{\cot A + \dfrac{\cot^2 A - 1}{2\cot A}} = \frac{\cot^3 A - \cot A - 2\cot A}{2\cot^2 A + \cot^2 A - 1}$$

$$= \frac{\cot^3 A - 3\cot A}{3\cot^2 A - 1} = \frac{3\cot A - \cot^3 A}{1 - 3\cot^2 A}$$

Another way of proving this result is

$$\cot 3A = \frac{1}{\tan 3A} = \frac{1 - 3\tan^2 A}{3\tan A - \tan^3 A} = \frac{1 - \dfrac{3}{\cot^2 A}}{\dfrac{3}{\cot A} - \dfrac{1}{\cot^3 A}} = \frac{\cot A\left(\cot^2 A - 3\right)}{3\cot^2 A - 1}$$

$$= \frac{3\cot A - \cot^3 A}{1 - 3\cot^2 A}.$$

6.3.6 Note

1. It can be verified independently that the formula for $\tan 3A$ given in Theorem 6.3.5 (iii) above, remains valid even when $2A$ is an odd multiple of $\dfrac{\pi}{2}$.

2. Also, the formula for $\cot 3A$ given in Theorem 6.3.5 (iv) above remains valid even if 2A is an integral multiple of π provided 3A is not an integral multiple of π. (That is, 2A is an odd integral multiple of π)

On replacing A by $\dfrac{A}{3}$ in the above Theorem 6.3.5, we get the following

6.3.7 Corollary

If A is any real number, then

(i) $\sin A = 3\sin\dfrac{A}{3} - 4\sin^3\dfrac{A}{3}$.

(ii) $\cos A = 4\cos^3\dfrac{A}{3} - 3\cos\dfrac{A}{3}$.

(iii) If A is not an odd multiple of $\dfrac{\pi}{2}$, then $\tan A = \dfrac{3\tan A/3 - \tan^3 A/3}{1 - 3\tan^2 A/3}$.

(iv) If A is not an integral multiple of π, then $\cot A = \dfrac{3\cot A/3 - \cot^3 A/3}{1 - 3\cot^2 A/3}$.

6.3.8 Theorem

For any $A \in \mathbf{R}$,

(i) $\sin A = \pm\sqrt{\dfrac{1-\cos 2A}{2}}$

(ii) $\cos A = \pm\sqrt{\dfrac{1+\cos 2A}{2}}$

(iii) *If* A *is not an odd multiple of* $\dfrac{\pi}{2}$, *then* $\tan A = \pm\sqrt{\dfrac{1-\cos 2A}{1+\cos 2A}}$

Proof

(i) We know that $\cos 2A = 1 - 2\sin^2 A$.

So $2\sin^2 A = 1 - \cos 2A$ and hence $\sin A = \pm\sqrt{\dfrac{1-\cos 2A}{2}}$.

(ii) $\cos 2A = 2\cos^2 A - 1 \Rightarrow 2\cos^2 A = 1 + \cos 2A \Rightarrow \cos A = \pm\sqrt{\dfrac{1+\cos 2A}{2}}$.

(iii) Assume that A is not an odd multiple of $\dfrac{\pi}{2}$. Then

$$\tan^2 A = \frac{2\sin^2 A}{2\cos^2 A} = \frac{1-\cos 2A}{1+\cos 2A} \Rightarrow \tan A = \pm\sqrt{\frac{1-\cos 2A}{1+\cos 2A}}.$$

On replacing A by $\dfrac{A}{2}$ in Theorem 6.3.7, we get the following.

6.3.9 Corollary

For any real number A,

(i) $\sin\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{2}}$

(ii) $\cos\dfrac{A}{2} = \pm\sqrt{\dfrac{1+\cos A}{2}}$

(iii) *If* A *is not an odd multiple of* π, *then* $\tan\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{1+\cos A}}$

6.3.10 Example

Prove that (i) $\sin 18^0 = \dfrac{\sqrt{5}-1}{4}$ (ii) $\cos 36^0 = \dfrac{\sqrt{5}+1}{4}$

Solution : (i) Write $A = 18^0$. Then $5A = 90^0$.

Now $2A = 90^0 - 3A \Rightarrow \sin 2A = \sin(90^0 - 3A) = \cos 3A$

$$\Rightarrow 2\sin A \cos A = 4\cos^3 A - 3\cos A$$

$$\Rightarrow 2\sin A = 4\cos^2 A - 3 \left(\text{since } \cos 18^0 \neq 0\right)$$

$$\Rightarrow 2\sin A = 4\left(1 - \sin^2 A\right) - 3$$

$$\Rightarrow 4\sin^2 A + 2\sin A - 1 = 0.$$

This is a quadratic equation in $\sin A$, so that

$$\sin A = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 + 16}}{8} = \frac{-2 \pm 2\sqrt{5}}{8}$$

$$= \frac{-1 \pm \sqrt{5}}{4}$$

Since A lies in first quadrant $\sin A > 0$.

Therefore $\sin A = \dfrac{\sqrt{5} - 1}{4}$. That is, $\sin 18^0 = \dfrac{\sqrt{5} - 1}{4}$.

(ii) $\cos 36^0 = \cos 2A$ (where $A = 18^0$)

$$= 1 - 2\sin^2 A = 1 - 2\left(\frac{\sqrt{5} - 1}{4}\right)^2 \text{ (from (i) above)}$$

$$= 1 - \frac{\left(6 - 2\sqrt{5}\right)}{8} = \frac{8 - 6 + 2\sqrt{5}}{8} = \frac{\sqrt{5} + 1}{4}.$$

6.3.11 Example

Find the values of (i) $\sin 36^0$ (ii) $\cos 18^0$

Solution: (i) $\sin 36^0 = \sqrt{1 - \cos^2 36^0}$ (we take positive square root because $\sin 36^0 > 0$)

$$= \sqrt{1 - \left(\frac{\sqrt{5} + 1}{4}\right)^2} = \sqrt{1 - \frac{\left(6 + 2\sqrt{5}\right)}{16}} = \frac{\sqrt{10 - 2\sqrt{5}}}{4}.$$

(ii) Similarly, we can prove that $\cos 18^0 = \sqrt{1 - \sin^2 18^0} = \dfrac{\sqrt{10 + 2\sqrt{5}}}{4}$.

6.3.12 Solved Problems

1. Problem: *Find the values of* (i) $\sin 22\frac{1}{2}^0$ (ii) $\cos 22\frac{1}{2}^0$ (iii) $\tan 22\frac{1}{2}^0$ (iv) $\cot 22\frac{1}{2}^0$.

Solution: If $A = 22\frac{1}{2}^0$, then $2A = 45^0$. Therefore, from Theorem 6.3.8, we get

(i) $\sin A = \sqrt{\frac{1 - \cos 2A}{2}}$ (since $\sin A > 0$)

$$\therefore \sin 22\frac{1}{2}^0 = \sqrt{\frac{1-\cos 45^0}{2}} = \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}.$$

(ii) $$\cos 22\frac{1^0}{2} = \sqrt{\frac{1+\cos 2A}{2}} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}.$$

(iii) $$\tan 22\frac{1}{2}^0 = \frac{\sin 22\frac{1}{2}^0}{\cos 22\frac{1}{2}^0} = \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} = \sqrt{\frac{\left(\sqrt{2}-1\right)^2}{2-1}} = \sqrt{2}-1.$$

(iv) $$\cot 22\frac{1}{2}^0 = \frac{1}{\tan 22\frac{1}{2}^0} = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1.$$

2. Problem: *Find the values of*

(i) $\sin 67\frac{1}{2}^0$ (ii) $\cos 67\frac{1}{2}^0$ (iii) $\tan 67\frac{1}{2}^0$ (iv) $\cot 67\frac{1}{2}^0$

Solution: This is a direct consequence of problem 1 above since $67\frac{1}{2}^0 = 90^0 - 22\frac{1}{2}^0$.

3. Problem: *Simplify* $\frac{1-\cos 2\theta}{\sin 2\theta}$.

Solution: $$\frac{1-\cos 2\theta}{\sin 2\theta} = \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

4. Problem: *If* $\cos A = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$, *find the value of* $\cos 2A$.

Solution: $$\cos 2A = 2\cos^2 A - 1 = 2\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right) - 1 = \frac{\sqrt{2}+1}{\sqrt{2}} - 1 = \frac{1}{\sqrt{2}}.$$

5. Problem: *If* $\cos\theta = \frac{-5}{13}$ *and* $\frac{\pi}{2} < \theta < \pi$, *find the value of* $\sin 2\theta$.

Solution: $\frac{\pi}{2} < \theta < \pi \Rightarrow \sin\theta > 0$ and $\cos\theta = -\frac{5}{13} \Rightarrow \sin\theta = \frac{12}{13}$

$$\therefore \sin 2\theta = 2\sin\theta\cos\theta$$

$$= 2\cdot\frac{12}{13}\left(\frac{-5}{13}\right) = \frac{-120}{169}.$$

6. Problem: *For what values of x in the first quadrant* $\frac{2\tan x}{1-\tan^2 x}$ *is positive ?*

Solution: $\frac{2\tan x}{1-\tan^2 x} > 0 \Rightarrow \tan 2x > 0$

$$\Rightarrow 0 < 2x < \frac{\pi}{2} \text{ (since } x \text{ is in the first quadrant)}$$

$$\Rightarrow 0 < x < \frac{\pi}{4}.$$

7. Problem: *If* $\cos\theta = \frac{-3}{5}$ *and* $\pi < \theta < \frac{3\pi}{2}$, *find the value of* $\tan \theta/2$.

Solution:

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \pm\sqrt{\frac{1+\frac{3}{5}}{1-\frac{3}{5}}} = \pm 2.$$

Given $\pi < \theta < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{\theta}{2} < \frac{3\pi}{4}$

$$\Rightarrow \tan\theta/2 < 0$$

$$\therefore \tan\theta/2 = -2.$$

8. Problem: *If A is not an integral multiple of* $\frac{\pi}{2}$, *prove that*

(i) $\tan A + \cot A = 2\operatorname{cosec} 2A$ (ii) $\cot A - \tan A = 2\cot 2A$

Solution: (i) $\tan A + \cot A = \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} = \frac{\sin^2 A + \cos^2 A}{\sin A \cos A}$

$$= \frac{1}{\sin A.\cos A} = \frac{2}{2\sin A\cos A} = \frac{2}{\sin 2A} = 2\operatorname{cosec} 2A.$$

(ii) $\cot A - \tan A = \frac{1}{\tan A} - \tan A = \frac{1-\tan^2 A}{\tan A} = 2\cot 2A$.

9. Problem: *If* θ *is not an integral multiple of* $\frac{\pi}{2}$, *prove that*

$$\tan\theta + 2\tan 2\theta + 4\tan 4\theta + 8\cot 8\theta = \cot\theta$$

Solution: From problem 8(ii) above $\tan A = \cot A - 2\cot 2A$ ... (1)

Therefore, $\tan\theta + 2\tan 2\theta + 4\tan 4\theta + 8\cot 8\theta$

$= (\cot\theta - 2\cot 2\theta) + 2(\cot 2\theta - 2\cot 4\theta) + 4(\cot 4\theta - 2\cot 8\theta) + 8\cot 8\theta$ (by (1) above)

$= \cot\theta$.

10. Problem: *For* $A \in \mathbf{R}$, *prove that*

(i) $\sin A \sin\left(\frac{\pi}{3} + A\right)\sin\left(\frac{\pi}{3} - A\right) = \frac{1}{4}\sin 3A$

(ii) $\cos A \cos\left(\frac{\pi}{3} + A\right)\cos\left(\frac{\pi}{3} - A\right) = \frac{1}{4}\cos 3A$ *and hence deduce that*

(iii) $\sin 20^0 \sin 40^0 \sin 60^0 \sin 80^0 = \frac{3}{16}$

(iv) $\cos\frac{\pi}{9} \cos\frac{2\pi}{9} \cos\frac{3\pi}{9} \cos\frac{4\pi}{9} = \frac{1}{16}$.

Solution

(i) $\sin A . \sin\left(\frac{\pi}{3} + A\right)\sin\left(\frac{\pi}{3} - A\right)$

$$= \sin A . \left(\sin^2\frac{\pi}{3} - \sin^2 A\right) \text{ (from Theorem 6.2.7 (i)).}$$

$$= \sin A\left(\frac{3}{4} - \sin^2 A\right) = \frac{\sin A\left(3 - 4\sin^2 A\right)}{4}$$

$$= \frac{1}{4}\left(3\sin A - 4\sin^3 A\right) = \frac{1}{4}\sin 3A.$$

(ii) $\cos A \cos\left(\frac{\pi}{3} + A\right)\cos\left(\frac{\pi}{3} - A\right)$

$$= \cos A\left(\cos^2 A - \sin^2\frac{\pi}{3}\right) \text{ (from Theorem 6.2.7 (ii))}$$

$$= \cos A\left(\cos^2 A - \frac{3}{4}\right) = \frac{\cos A\left(4\cos^2 A - 3\right)}{4}$$

$$= \frac{1}{4}\left(4\cos^3 A - 3\cos A\right) = \frac{1}{4}\cos 3A.$$

(iii) Substituting $A = 20^0$ in (i) above, we get

$$\sin 20^0 \sin\left(60^0 + 20^0\right)\sin\left(60^0 - 20^0\right) = \frac{1}{4}\sin 3\left(20^0\right)$$

$$\sin 20^0 \sin 80^0 \sin 40^0 = \frac{1}{4}\sin 60^0$$

On multiplying both sides of the above equation by $\sin 60^0$, we get

$$\sin 20^0 \sin 40^0 \sin 60^0 \sin 80^0 = \frac{1}{4}\sin^2 60^0 = \frac{1}{4}.\frac{3}{4} = \frac{3}{16}$$

Similarly, from (ii) above, we get

(iv) $\cos 20^0 \cos 40^0 \cos 80^0 = \frac{1}{4}\cos 60^0$

On multiplying both sides by $\cos 60^0$, we get

$$\cos 20^0 \cos 40^0 \cos 60^0 \cos 80^0 = \frac{1}{4}\cos^2 60^0 = \frac{1}{16}$$

That is, $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{3\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{16}$.

11. Problem: *If 3A is not an odd multiple of* $\frac{\pi}{2}$, *prove that*

$$\tan A \,.\, \tan\left(60^0 + A\right) . \tan\left(60^0 - A\right) = \tan 3A$$

and hence find the value of $\tan 6^0 \tan 42^0 \tan 66^0 \tan 78^0$.

Solution: From problems 10(i) and 10(ii) above, we have

$$\sin A . \sin\left(60^0 + A\right) . \sin\left(60^0 - A\right) = \frac{1}{4}\sin 3A \quad \text{... (1)}$$

$$\cos A . \cos\left(60^0 + A\right) . \cos\left(60^0 - A\right) = \frac{1}{4}\cos 3A \quad \text{... (2)}$$

On dividing (1) and (2), we get

$$\tan A . \tan\left(60^0 + A\right) . \tan\left(60^0 - A\right) = \tan 3A \quad \text{... (3)}$$

Now, put $A = 6^0$ in (3), we get

$$\tan 6^0 . \tan 66^0 . \tan 54^0 = \tan 18^0 \quad \text{... (4)}$$

Again on substituting $A = 18^0$ in (3) above, we get

$$\tan 18^0 . \tan 78^0 . \tan 42^0 = \tan 54^0 \quad \text{... (5)}$$

On multiplying (4) and (5), we get

$$\left(\tan 6^0 . \tan 66^0 . \tan 54^0\right) . \left(\tan 18^0 . \tan 78^0 . \tan 42^0\right) = \tan 18^0 . \tan 54^0$$

Hence, we get $\tan 6^0 . \tan 42^0 . \tan 66^0 . \tan 78^0 = 1$.

12. Problem: *For* $\alpha, \beta \in \mathbf{R}$, *prove that* $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\frac{(\alpha - \beta)}{2}$.

Solution: $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2$

$$= \cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta + \sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta$$

$$= 2 + 2\left(\cos\alpha\,\cos\beta + \sin\alpha\,\sin\beta\right)\left(\text{since } \cos^2\alpha + \sin^2\alpha = 1 = \cos^2\beta + \sin^2\beta\right)$$

$$= 2\left(1+\cos\left(\alpha-\beta\right)\right)$$

$$= 2.\ 2\cos^2\left(\frac{\alpha-\beta}{2}\right) \quad \text{(by from 6.3.2 (ii))}$$

$$= 4\cos^2\frac{\alpha-\beta}{2}.$$

13. Problem: *If a, b, c are nonzero real numbers and* $\alpha,\ \beta$ *are solutions of the equation* $a\cos\theta + b\sin\theta = c$ *then show that (i)* $\sin\alpha + \sin\beta = \frac{2bc}{a^2+b^2}$ *and (ii)* $\sin\alpha \,.\, \sin\beta = \frac{c^2-a^2}{a^2+b^2}$.

Solution: $a\cos\theta + b\sin\theta = c$

$$\Rightarrow a\cos\theta = c - b\sin\theta$$

$$\Rightarrow a^2\cos^2\theta = c^2 - 2bc\sin\theta + b^2\sin^2\theta$$

$$\Rightarrow a^2\left(1-\sin^2\theta\right) = c^2 - 2bc\sin\theta + b^2\sin^2\theta$$

$$\Rightarrow \left(a^2+b^2\right)\sin^2\theta - 2bc\sin\theta + \left(c^2-a^2\right) = 0\,.$$

This is a quadratic equation in $\sin\theta$, whose roots are $\sin\alpha$ and $\sin\beta$ (since $\alpha,\ \beta$ are two solutions of the given equation). Therefore,

$$\sin\alpha + \sin\beta = \text{sum of the roots} = \frac{2bc}{a^2+b^2}$$

$$\sin\alpha \,.\, \sin\beta = \text{product of the roots} = \frac{c^2-a^2}{a^2+b^2}\,.$$

14. Problem: *If* θ *is not an odd multiple of* $\frac{\pi}{2}$ *and* $\cos\theta \neq -\frac{1}{2}$, *prove that*

$$\frac{\sin\theta + \sin 2\theta}{1+\cos\theta+\cos 2\theta} = \tan\theta.$$

Solution: $\frac{\sin\theta + \sin 2\theta}{1+\cos\theta+\cos 2\theta} = \frac{\sin\theta + 2\sin\theta\cos\theta}{\cos\theta + 2\cos^2\theta}$ (from Theorem 6.3.2)

$$= \frac{\sin\theta\left(1+2\cos\theta\right)}{\cos\theta\left(1+2\cos\theta\right)}$$

$$= \frac{\sin\theta}{\cos\theta}\left(\text{since}\cos\theta \neq -\frac{1}{2}\right) = \tan\theta.$$

15. Problem: *Prove that* $\sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{5\pi}{8} + \sin^4\frac{7\pi}{8} = \frac{3}{2}$.

Solution: $\sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{5\pi}{8} + \sin^4\frac{7\pi}{8}$

$$= \sin^4\frac{\pi}{8} + \sin^4\left(\frac{\pi}{2}-\frac{\pi}{8}\right) + \sin^4\left(\frac{\pi}{2}+\frac{\pi}{8}\right) + \sin^4\left(\pi-\frac{\pi}{8}\right)$$

$$= \sin^4\frac{\pi}{8} + \cos^4\frac{\pi}{8} + \cos^4\frac{\pi}{8} + \sin^4\frac{\pi}{8}$$

$$= 2\left\{\sin^4\frac{\pi}{8} + \cos^4\frac{\pi}{8}\right\}$$

$$= 2\left\{\left(\sin^2\frac{\pi}{8} + \cos^2\frac{\pi}{8}\right)^2 - 2\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}\right\} \left(\text{since } a^2+b^2=(a+b)^2-2ab\right)$$

$$= 2\left(1 - 2\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}\right)$$

$$= 2 - 4\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8} = 2 - \left(2\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right)^2$$

$$= 2 - \left(\sin\frac{\pi}{4}\right)^2 = 2 - \frac{1}{2} = \frac{3}{2}.$$

16. Problem: *If none of 2A and 3A is an odd multiple of $\pi/2$, then prove that*

$$\tan 3A \tan 2A \tan A = \tan 3A - \tan 2A - \tan A.$$

Solution: $\tan 3A = \tan(2A + A) = \dfrac{\tan 2A + \tan A}{1 - \tan A . \tan 2A}$

$$\Rightarrow \tan 3A(1 - \tan A \tan 2A) = \tan 2A + \tan A$$

$$\Rightarrow \tan 3A - \tan A \tan 2A \tan 3A = \tan 2A + \tan A$$

$$\Rightarrow \tan 3A - \tan 2A - \tan A = \tan A \tan 2A \tan 3A.$$

Exercise 6(d)

I. 1. Simplify (i) $\dfrac{\sin 2\theta}{1+\cos 2\theta}$ (ii) $\dfrac{3\cos\theta + \cos 3\theta}{3\sin\theta - \sin 3\theta}$

2. Evaluate the following

(i) $6 \sin 20^0 - 8 \sin^3 20^0$ (ii) $\cos^2 72^0 - \sin^2 54^0$ (iii) $\sin^2 42^0 - \sin^2 12^0$

3. (i) Express $\dfrac{\sin 4\theta}{\sin\theta}$ in terms of $\cos^3\theta$ and $\cos\theta$.

(ii) Express $\cos^6 A + \sin^6 A$ in terms of $\sin 2A$.

(iii) Express $\dfrac{1-\cos\theta+\sin\theta}{1+\cos\theta+\sin\theta}$ in terms of $\tan\theta/2$.

4. (i) If $\sin\alpha = \frac{3}{5}$, where $\frac{\pi}{2} < \alpha < \pi$, evaluate cos 3α and tan 2α.

(ii) If $\cos A = \frac{7}{25}$ and $\frac{3\pi}{2} < A < 2\pi$, then find the value of cot A/2.

(iii) If $0 < \theta < \frac{\pi}{8}$, show that $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 4\theta}}} = 2\cos(\theta/2)\cdot$

5. Find the extreme values of (i) $\cos 2x + \cos^2 x$. (ii) $3\sin^2 x + 5\cos^2 x$

6. If $a \le \cos\theta + 3\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) + 6 \le b$, then find the largest value of a and smallest value of b.

7. Find the periods for the following functions

(i) $\cos^4 x$ (ii) $2\sin\left(\frac{\pi x}{4}\right) + 3\cos\left(\frac{\pi x}{3}\right)$ (iii) $\sin^2 x + 2\cos^2 x$

(iv) $2\sin\left(\frac{\pi}{4} + x\right)\cos x$ (v) $\frac{5\sin x + 3\cos x}{4\sin 2x + 5\cos x}$

II.

1. (i) If $0 < A < \frac{\pi}{4}$, and $\cos A = \frac{4}{5}$, find the values of sin 2A and cos 2A.

(ii) For what values of A in the first quadrant, the expression $\frac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$ is positive ?

(iii) Prove that $\frac{\cos 3A + \sin 3A}{\cos A - \sin A} = 1 + 2\sin 2A$.

2. (i) Prove that $\cot\left(\frac{\pi}{4} - \theta\right) = \frac{\cos 2\theta}{1 - \sin 2\theta}$ and hence find the value of $\cot 15^0$.

(ii) If θ lies in third quadrant and $\sin\theta = \frac{-4}{5}$, find the values of cosec (θ/2) and tan (θ/2.)

(iii) If $450^0 < \theta < 540^0$ and $\sin\theta = \frac{12}{13}$, then calculate sin (θ /2) and cos (θ /2).

(iv) Prove that $\frac{1}{\cos 290^0} + \frac{1}{\sqrt{3}\sin 250^0} = \frac{4}{\sqrt{3}}$.

3. Prove that

(i) $\frac{\sin 2A}{(1 - \cos 2A)} \cdot \frac{(1 - \cos A)}{\cos A} = \tan\frac{A}{2}$.

(ii) $\frac{\sin 2x}{(\sec x + 1)} \cdot \frac{\sec 2x}{(\sec 2x + 1)} = \tan\left(\frac{x}{2}\right)$.

(iii) $\frac{(\cos^3\theta - \cos 3\theta)}{\cos\theta} + \frac{(\sin^3\theta + \sin 3\theta)}{\sin\theta} = 3$.

4. (i) Show that $\cos A = \dfrac{\cos 3A}{(2\cos 2A - 1)}$. Hence find the value of cos 15^0.

(ii) Show that $\sin A = \dfrac{\sin 3A}{1 + 2\cos 2A}$. Hence find the value of sin 15^0.

(iii) Prove that $\tan\alpha = \dfrac{\sin 2\alpha}{1+\cos 2\alpha}$ and hence deduce the values of $\tan 15^0$ and $\tan 22\frac{1}{2}^0$.

5. Prove that

(i) $\dfrac{1}{\sin 10^0} - \dfrac{\sqrt{3}}{\cos 10^0} = 4$

(ii) $\sqrt{3}\,\text{cosec}\ 20^0 - \sec 20^0 = 4$

(iii) $\tan 9^0 - \tan 27^0 - \cot 27^0 + \cot 9^0 = 4$.

(iv) If $\dfrac{\sin\alpha}{a} = \dfrac{\cos\alpha}{b}$, then prove that $a \sin 2\alpha + b\cos 2\alpha = b$.

6. (i) In a ΔABC; if $\tan\dfrac{A}{2} = \dfrac{5}{6}$ and $\tan\dfrac{B}{2} = \dfrac{20}{37}$, then show that $\tan\left(\dfrac{C}{2}\right) = \dfrac{2}{5}$.

(ii) If $\cos\theta = \dfrac{5}{13}$ and $270^0 < \theta < 360^0$, evaluate $\sin(\theta/2)$ and $\cos(\theta/2)$.

(iii) If $180^0 < \theta < 270^0$ and $\sin\theta = \dfrac{-4}{5}$ calculate $\sin\left(\dfrac{\theta}{2}\right)$ and $\cos\left(\dfrac{\theta}{2}\right)$.

7. (i) Prove that $\cos^2\dfrac{\pi}{8} + \cos^2\dfrac{3\pi}{8} + \cos^2\dfrac{5\pi}{8} + \cos^2\dfrac{7\pi}{8} = 2$.

(ii) Show that $\cos^4\left(\dfrac{\pi}{8}\right) + \cos^4\left(\dfrac{3\pi}{8}\right) + \cos^4\left(\dfrac{5\pi}{8}\right) + \cos^4\left(\dfrac{7\pi}{8}\right) = \dfrac{3}{2}$.

III.

1. (i) If $\tan x + \tan\left(x + \dfrac{\pi}{3}\right) + \tan\left(x + \dfrac{2\pi}{3}\right) = 3$, show that $\tan 3x = 1$.

(ii) Prove that $\sin\dfrac{\pi}{5}.\sin\dfrac{2\pi}{5}.\sin\dfrac{3\pi}{5}.\sin\dfrac{4\pi}{5} = \dfrac{5}{16}$.

(iii) Show that $\cos^2\left(\dfrac{\pi}{10}\right) + \cos^2\left(\dfrac{2\pi}{5}\right) + \cos^2\left(\dfrac{3\pi}{5}\right) + \cos^2\left(\dfrac{9\pi}{10}\right) = 2$.

2. Prove that

(i) $\frac{1-\sec 8\alpha}{1-\sec 4\alpha} = \frac{\tan 8\alpha}{\tan 2\alpha}$

(ii) $\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right) = \frac{1}{16}$.

3. Prove that

(i) $\cos\frac{2\pi}{7}.\cos\frac{4\pi}{7}.\cos\frac{8\pi}{7} = \frac{1}{8}$.

(ii) $\cos\frac{\pi}{11}.\cos\frac{2\pi}{11}.\cos\frac{3\pi}{11}.\cos\frac{4\pi}{11}.\cos\frac{5\pi}{11} = \frac{1}{32}$

4. (i) If $\cos\alpha = \frac{3}{5}$ and $\cos\beta = \frac{5}{13}$ and α, β are acute angles, then prove that

(a) $\sin^2\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{65}$ and (b) $\cos^2\left(\frac{\alpha+\beta}{2}\right) = \frac{16}{65}$

(ii) If A is not an integral multiple of π, prove that

$\cos A.\ \cos 2A.\ \cos 4A.\ \cos 8A = \frac{\sin 16A}{16\sin A}$ and hence deduce that

$\cos\frac{2\pi}{15}.\cos\frac{4\pi}{15}.\cos\frac{8\pi}{15}.\cos\frac{16\pi}{15} = \frac{1}{16}$.

6.4 Sum and product transformations

Making use of the formulae of $\sin(A+B)$, $\sin(A-B)$, $\cos(A+B)$, $\cos(A-B)$ etc., in this section we give formulae to transform the sum (difference) of two trigonometric ratios into products and vice-versa.

6.4.1 Theorem (Transformation of product into sum)

For $A, B \in \mathbf{R}$ *we have*

1. $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
2. $2\cos A\sin B = \sin(A+B) - \sin(A-B)$
3. $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
4. $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Proof : We know that, $\sin(A+B) = \sin A\cos B + \cos A\sin B$ and

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

On adding these identities, we get

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B. \quad \text{... (1)}$$

On subtracting, we get

$$\sin(A+B)-\sin(A-B)=2\cos A\sin B. \quad \text{... (2)}$$

Similarly, we have

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$

$$\cos(A-B)=\cos A\cos B+\sin A\sin B.$$

On adding, these two identities, we get

$$\cos(A+B)+\cos(A-B)=2\cos A\cos B. \quad \text{... (3)}$$

On subtracting, we get

$$\cos(A+B)-\cos(A-B)=-2\sin A\sin B \text{ (or)}$$

$$\cos(A-B)-\cos(A+B)=2\sin A\sin B. \quad \text{... (4)}$$

6.4.2 Note

The four identities in the above theorem can be remembered easily as follows

$$2\sin A\cos B = \sin(\text{sum})+\sin(\text{difference})$$

$$2\cos A\sin B = \sin(\text{sum})-\sin(\text{difference})$$

$$2\cos A\cos B = \cos(\text{sum})+\cos(\text{difference})$$

$$2\sin A\sin B = \cos(\text{difference})-\cos(\text{sum})$$

In the following theorem we give **transformations from sum into products**.

6.4.3 Theorem

For any two real numbers C and D, we have

1. $\sin C+\sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

2. $\sin C-\sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

3. $\cos C+\cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

4. $\cos C-\cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

Proof : Write $A=\frac{C+D}{2}$ and $B=\frac{C-D}{2}$. Then $A+B=C$ and $A-B=D$.

Now, we get, from Theorem 6.4.1, all the above 4 transformations.

6.4.4 Solved Problems

1. Problem: *Prove that* $\sin 78^0 + \cos 132^0 = \dfrac{\sqrt{5}-1}{4}$.

Solution: $\sin 78^0 + \cos 132^0 = \sin 78^0 + \cos\left(90^0 + 42^0\right)$

$$= \sin 78^0 - \sin 42^0 = 2\cos\frac{78^0+42^0}{2}\sin\frac{78^0-42^0}{2}$$

$$= 2\cos 60^0 \sin 18^0 = 2\cdot\frac{1}{2}\cdot\frac{\sqrt{5}-1}{4} = \frac{\sqrt{5}-1}{4}.$$

2. Problem: *Prove that* $\sin 21^0\cos 9^0 - \cos 84^0\cos 6^0 = \dfrac{1}{4}$.

Solution: $\sin 21^0\cos 9^0 - \cos 84^0\cos 6^0$

$$= \frac{1}{2}\left(2\sin 21^0\cos 9^0 - 2\cos\left(90^0 - 6^0\right)\cos 6^0\right)$$

$$= \frac{1}{2}\left(\sin\left(21^0 + 9^0\right) + \sin\left(21^0 - 9^0\right) - 2\sin 6^0\cos 6^0\right)$$

$$= \frac{1}{2}\left(\sin 30^0 + \sin 12^0 - \sin 12^0\right) = \frac{1}{2}\sin 30^0 = \frac{1}{4}.$$

3. Problem: *Find the value of* $\sin 34^0 + \cos 64^0 - \cos 4^0$.

Solution: $\sin 34^0 + \left(\cos 64^0 - \cos 4^0\right)$

$$= \sin 34^0 - 2\sin\frac{64^0+4^0}{2}\sin\frac{64^0-4^0}{2} \quad \text{(by Theorem 6.4.3 (4))}$$

$$= \sin 34^0 - 2.\sin 34^0.\sin 30^0$$

$$= 0 \ \left(\text{since } \sin 30^0 = \frac{1}{2}\right).$$

4. Problem: *Prove that* $\cos^2 76^0 + \cos^2 16^0 - \cos 76^0\cos 16^0 = \dfrac{3}{4}$.

Solution: $\cos^2 76^0 + \cos^2 16^0 - \cos 76^0\cos 16^0$

$$= \cos^2 76^0 + \left(1 - \sin^2 16^0\right) - \frac{1}{2}\left(2\cos 76^0\cos 16^0\right)$$

$$= 1 + \left(\cos^2 76^0 - \sin^2 16^0\right) - \frac{1}{2}\left(\cos\left(76^0 + 16^0\right) + \cos\left(76^0 - 16^0\right)\right)$$

$$= 1 + \cos\left(76^0 + 16^0\right)\cos\left(76^0 - 16^0\right) - \frac{1}{2}\left(\cos 92^0 + \cos 60^0\right)$$

$$= 1 + \cos 92^0\cos 60^0 - \frac{1}{2}\cos 92^0 - \frac{1}{2}\cos 60^0$$

$$= 1 + \frac{1}{2}\cos 92^0 - \frac{1}{2}\cos 92^0 - \frac{1}{4} \qquad (\text{since } \cos 60^0 = \frac{1}{2})$$

$$= \frac{3}{4}.$$

5. Problem: *If* $a, b \neq 0$ *and* $\sin x + \sin y = a$ *and* $\cos x + \cos y = b$, *find the values of*

(i) $\tan\frac{x+y}{2}$, (ii) $\sin\frac{x-y}{2}$ *interms of* a *and* b.

Solution

(i) $\sin x + \sin y = a \Rightarrow 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = a$ ... (1)

$\cos x + \cos y = b \Rightarrow 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = b$ ... (2)

On dividing (1) by (2), we get $\tan\frac{x+y}{2} = \frac{a}{b}$.

(ii) First method

$$a^2 + b^2 = (\sin x + \sin y)^2 + (\cos x + \cos y)^2$$

$$= \sin^2 x + \sin^2 y + 2\sin x \sin y + \cos^2 x + \cos^2 y + 2\cos x\cos y$$

$$= 2 + 2(\cos x\cos y + \sin x\sin y)$$

$$= 2 + 2\cos(x - y)$$

$$\therefore \cos(x - y) = \frac{a^2 + b^2 - 2}{2}$$

Now $2\sin^2\left(\frac{x-y}{2}\right) = 1 - \cos(x-y) = 1 - \frac{a^2 + b^2 - 2}{2} = \frac{4 - a^2 - b^2}{2}$

$$\Rightarrow \sin\left(\frac{x-y}{2}\right) = \pm\frac{\sqrt{4 - a^2 - b^2}}{2}.$$

Second method

From (1) and (2), we get

$$a^2 + b^2 = 4\sin^2\left(\frac{x+y}{2}\right)\cos^2\left(\frac{x-y}{2}\right) + 4\cos^2\left(\frac{x+y}{2}\right)\cos^2\left(\frac{x-y}{2}\right)$$

$$= 4\cos^2\left(\frac{x-y}{2}\right)\left\{\sin^2\frac{x+y}{2} + \cos^2\frac{x+y}{2}\right\}$$

$$= 4\cos^2\left(\frac{x-y}{2}\right)$$

$$\Rightarrow \cos^2\left(\frac{x-y}{2}\right) = \frac{a^2 + b^2}{4}.$$

Now, $\sin^2\left(\frac{x-y}{2}\right) = 1-\cos^2\left(\frac{x-y}{2}\right) = 1-\frac{a^2+b^2}{4} = \frac{4-a^2-b^2}{4}$

Hence $\sin\left(\frac{x-y}{2}\right) = \pm\frac{\sqrt{4-a^2-b^2}}{2}$.

6. Problem: *Prove that* $\cos 12^0 + \cos 84^0 + \cos 132^0 + \cos 156^0 = -\frac{1}{2}$.

Solution: $\cos 12^0 + \cos 84^0 + \cos 132^0 + \cos 156^0$

$$= \left(\cos 12^0 + \cos 132^0\right) + \left(\cos 84^0 + \cos 156^0\right)$$

$$= 2\cos\frac{132^0+12^0}{2}.\cos\frac{132^0-12^0}{2} + 2\cos\frac{84^0+156^0}{2}\cos\frac{156^0-84^0}{2}$$

$$= 2\cos 72^0.\cos 60^0 + 2\cos 120^0\cos 36^0$$

$$= 2.\sin 18^0.\frac{1}{2} + 2\left(-\frac{1}{2}\right)\cos 36^0$$

$$= \sin 18^0 - \cos 36^0 = \left(\frac{\sqrt{5}-1}{4}\right) - \left(\frac{\sqrt{5}+1}{4}\right) = -\frac{1}{2}.$$

7. Problem: *Show that, for any* $\theta \in \mathbf{R}$,

$$4\sin\frac{5\theta}{2}\cos\frac{3\theta}{2}\cos 3\theta = \sin\theta - \sin 2\theta + \sin 4\theta + \sin 7\theta.$$

Solution: $4\sin\frac{5\theta}{2}\cos\frac{3\theta}{2}\cos 3\theta = 2\left(2\sin\frac{5\theta}{2}\cos\frac{3\theta}{2}\right)\cos 3\theta$

$$= 2\left(\sin 4\theta + \sin\theta\right)\cos 3\theta$$

$$= 2\sin 4\theta\cos 3\theta + 2\cos 3\theta\sin\theta$$

$$= \sin\left(4\theta + 3\theta\right) + \sin\left(4\theta - 3\theta\right) + \sin\left(3\theta + \theta\right) - \sin\left(3\theta - \theta\right)$$

$$= \sin 7\theta + \sin\theta + \sin 4\theta - \sin 2\theta.$$

8. Problem: *If none of* A, B, A + B *is an integral multiple of* π, *then*

prove that $\frac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)} = \tan\frac{A}{2}\cot\frac{B}{2}$.

Solution : $1-\cos A+\cos B-\cos(A+B) = 1-\cos(A+B) - (\cos A - \cos B)$

$$= 2\sin^2\frac{A+B}{2} + 2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$= 2\sin\frac{A+B}{2}\left\{\sin\frac{A+B}{2} + \sin\frac{A-B}{2}\right\}$$

$$= 2\sin\frac{A+B}{2}\left\{2\sin\frac{A}{2}\cos\frac{B}{2}\right\} \qquad \dots (1)$$

Now, $1+\cos A - \cos B - \cos(A+B)$

$$= \left(1-\cos(A+B)\right) + \left(\cos A - \cos B\right)$$

$$= 2\sin^2\frac{A+B}{2} - 2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$= 2\sin\frac{A+B}{2}\left\{\sin\frac{A+B}{2} - \sin\frac{A-B}{2}\right\}$$

$$= 2\sin\left(\frac{A+B}{2}\right)\left(2\cos\frac{A}{2}\sin\frac{B}{2}\right) \qquad \dots (2)$$

From (1) and (2), we get that

$$\frac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)} = \frac{4\sin\left(\frac{A+B}{2}\right).\sin\frac{A}{2}\cos\frac{B}{2}}{4\sin\left(\frac{A+B}{2}\right)\cos\frac{A}{2}\sin\frac{B}{2}}$$

$$= \tan\frac{A}{2}\cot\frac{B}{2}.$$

9. Problem: *For any* $\alpha \in \mathbf{R}$, *prove that* $\cos^2\left(\alpha-\frac{\pi}{4}\right)+\cos^2\left(\alpha+\frac{\pi}{12}\right)-\cos^2\left(\alpha-\frac{\pi}{12}\right)=\frac{1}{2}$.

Solution: $\cos^2\left(\alpha-\frac{\pi}{4}\right)+\left\{\cos^2\left(\alpha+\frac{\pi}{12}\right)-\cos^2\left(\alpha-\frac{\pi}{12}\right)\right\}$

$$= \frac{1+\cos\left(2\alpha-\frac{\pi}{2}\right)}{2} + \sin\left(\left(\alpha-\frac{\pi}{12}\right)+\left(\alpha+\frac{\pi}{12}\right)\right)\cdot\sin\left(\left(\alpha-\frac{\pi}{12}\right)-\left(\alpha+\frac{\pi}{12}\right)\right)$$

$$= \frac{1+\sin 2\alpha}{2} + \sin 2\alpha.\sin\left(-\frac{\pi}{6}\right) = \frac{1}{2}+\frac{1}{2}\sin 2\alpha - \frac{1}{2}\sin 2\alpha = \frac{1}{2}.$$

10. Problem: *Suppose* $(\alpha-\beta)$ *is not an odd multiple of* $\frac{\pi}{2}$, m *is a non zero real number such that* $m \neq -1$ *and* $\frac{\sin(\alpha+\beta)}{\cos(\alpha-\beta)} = \frac{1-m}{1+m}$. *Then prove that* $\tan\left(\frac{\pi}{4}-\alpha\right) = m\,.\tan\left(\frac{\pi}{4}+\beta\right)$.

Solution: Given that $\frac{\sin(\alpha+\beta)}{\cos(\alpha-\beta)} = \frac{1-m}{1+m}$.

By using componendo and dividendo, we get

$$\frac{\sin(\alpha+\beta)+\cos(\alpha-\beta)}{\sin(\alpha+\beta)-\cos(\alpha-\beta)}=\frac{1-m+1+m}{(1-m)-(1+m)}=\frac{2}{-2m}=\frac{-1}{m}\text{ (given that } m\neq 0)$$

$$\Rightarrow m\{\sin(\alpha+\beta)+\cos(\alpha-\beta)\}=-\{\sin(\alpha+\beta)-\cos(\alpha-\beta)\}$$

$$\Rightarrow m\left\{\sin(\alpha+\beta)+\sin\left(\frac{\pi}{2}-(\alpha-\beta)\right)\right\}$$

$$=-\left\{\sin(\alpha+\beta)-\sin\left(\frac{\pi}{2}-(\alpha-\beta)\right)\right\}$$

$$\Rightarrow m.2\sin\left(\frac{\alpha+\beta+\frac{\pi}{2}-\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta-\frac{\pi}{2}+\alpha-\beta}{2}\right)$$

$$=-2\cos\left(\frac{\alpha+\beta+\frac{\pi}{2}-\alpha+\beta}{2}\right)\sin\left(\frac{\alpha+\beta-\frac{\pi}{2}+\alpha-\beta}{2}\right)$$

$$\Rightarrow m.\sin\left(\frac{\pi}{4}+\beta\right).\cos\left(\alpha-\frac{\pi}{4}\right)=-\cos\left(\frac{\pi}{4}+\beta\right)\sin\left(\alpha-\frac{\pi}{4}\right)$$

$$\Rightarrow m\sin\left(\frac{\pi}{4}+\beta\right).\cos\left(\frac{\pi}{4}-\alpha\right)=\cos\left(\frac{\pi}{4}+\beta\right)\sin\left(\frac{\pi}{4}-\alpha\right)$$

(since $\cos(-\theta)=\cos\theta$ *and* $\sin(-\theta)=-\sin\theta$)

$$\Rightarrow m.\frac{\sin\left(\frac{\pi}{4}+\beta\right)}{\cos\left(\frac{\pi}{4}+\beta\right)}=\frac{\sin\left(\frac{\pi}{4}-\alpha\right)}{\cos\left(\frac{\pi}{4}-\alpha\right)}$$

$$\Rightarrow m\tan\left(\frac{\pi}{4}+\beta\right)=\tan\left(\frac{\pi}{4}-\alpha\right).$$

Exercise 6(e)

I. 1. Prove that $\sin 50^0-\sin 70^0+\sin 10^0=0$.

2. Prove that $\dfrac{\sin 70^0-\cos 40^0}{\cos 50^0-\sin 20^0}=\dfrac{1}{\sqrt{3}}$.

3. Prove that $\cos 55^0 + \cos 65^0 + \cos 175^0 = 0$.

4. Prove that $4\left(\cos 66^0 + \sin 84^0\right) = \sqrt{3} + \sqrt{15}$.

5. Prove that $\cos 20^0 \cos 40^0 - \sin 5^0 \sin 25^0 = \dfrac{\sqrt{3}+1}{4}$.

6. Prove that $\cos 48^0 . \cos 12^0 = \dfrac{3+\sqrt{5}}{8}$.

II. 1. Prove that $\cos\theta + \cos\left(\dfrac{2\pi}{3} + \theta\right) + \cos\left(\dfrac{4\pi}{3} + \theta\right) = 0$.

2. Prove that $\sin^2\left(\alpha - \dfrac{\pi}{4}\right) + \sin^2\left(\alpha + \dfrac{\pi}{12}\right) - \sin^2\left(\alpha - \dfrac{\pi}{12}\right) = \dfrac{1}{2}$.

3. If $\sin x + \sin y = \dfrac{1}{4}$ and $\cos x + \cos y = \dfrac{1}{3}$, then show that

(i) $\tan\dfrac{x+y}{2} = \dfrac{3}{4}$ (ii) $\cot(x+y) = \dfrac{7}{24}$

4. If neither $\left(A - \dfrac{\pi}{12}\right)$ nor $\left(A - \dfrac{5\pi}{12}\right)$ is an integral multiple of π, prove that

$$\cot\left(\frac{\pi}{12} - A\right) + \tan\left(\frac{\pi}{12} + A\right) = \frac{4\cos 2A}{1 - 2\sin 2A}.$$

5. Prove that $4\cos 12^0 \cos 48^0 \cos 72^0 = \cos 36^0$.

6. Prove that $\sin 10^0 + \sin 20^0 + \sin 40^0 + \sin 50^0 = \sin 70^0 + \sin 80^0$.

III. 1. If $\cos x + \cos y = \dfrac{4}{5}$ and $\cos x - \cos y = \dfrac{2}{7}$, find the value of $14\tan\dfrac{x-y}{2} + 5\cot\dfrac{x+y}{2}$.

2. If none of the denominators is zero, prove that

$$\left(\frac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\frac{\sin A + \sin B}{\cos A - \cos B}\right)^n = \begin{cases} 2.\cot^n\left(\dfrac{A-B}{2}\right), & \textit{if } n \textit{ is even} \\ 0, & \textit{if } n \textit{ is odd} \end{cases}$$

3. If $\sin A = \sin B$ and $\cos A = \cos B$, then prove that $A = 2n\pi + B$ for some integer n.

4. If $\cos n\alpha \neq 0$ and $\cos\dfrac{\alpha}{2} \neq 0$, then show that

$$\frac{\sin(n+1)\alpha - \sin(n-1)\alpha}{\cos(n+1)\alpha + 2\cos n\alpha + \cos(n-1)\alpha} = \tan\frac{\alpha}{2}.$$

5. If $\sec(\theta + \alpha) + \sec(\theta - \alpha) = 2\sec\theta$ and $\cos\alpha \neq 1$, then show that $\cos\theta = \pm\sqrt{2}\cos\dfrac{\alpha}{2}$.

6. If none of x, y, z is an odd multiple of $\frac{\pi}{2}$ and if $\sin(y+z-x)$, $\sin(z+x-y)$, $\sin(x+y-z)$ are in A.P., then prove that $\tan x$, $\tan y$, $\tan z$ are also in A.P.

7. If x, y, z are non zero real numbers and if

$$x\cos\theta = y\cos\left(\theta + \frac{2\pi}{3}\right) = z\cos\left(\theta + \frac{4\pi}{3}\right) \text{ for some } \theta \in \mathbf{R}\text{, then}$$

show that $xy + yz + zx = 0$.

8. If neither A nor A + B is an odd multiple of $\frac{\pi}{2}$ and if $m\sin B = n\sin(2A + B)$, then prove that $(m+n)\tan A = (m-n)\tan(A+B)$.

9. If $\tan(A+B) = \lambda\tan(A-B)$, then show that $(\lambda+1)\sin 2B = (\lambda-1)\sin 2A$.

6.4.5 Identities

When A, B, C are 3 real numbers satisfying the conditions like $A + B + C = 0$ or $A + B + C = \frac{\pi}{2}$ or $A + B + C = \pi$ or $A + B + C = 2\pi$ etc., we prove some identities relating to the trigonometric ratios of A, B, C.

1. If A, B, C are angles in a triangle, that is, if $A + B + C = \pi$, then we have the following identities.

$$A + B + C = \pi \quad \Rightarrow \quad A + B = \pi - C$$

So that, $\sin(A+B) = \sin(180^0 - C) = \sin C$. Similarly,

$$\sin(B+C) = \sin A \text{ and } \sin(C+A) = \sin B.$$

Also $\cos(A+B) = \cos(\pi - C) = -\cos C$. Similarly,

$$\cos(B+C) = -\cos A \text{ and } \cos(C+A) = -\cos B.$$

2. If $A + B + C = \frac{\pi}{2}$, then we have

$$\sin(A+B) = \sin\left(\frac{\pi}{2} - C\right) = \cos C\text{, similarly, we get}$$

$$\sin(B+C) = \cos A;\ \sin(C+A) = \cos B;\ \cos(A+B) = \sin C;$$

$$\cos(B+C) = \sin A;\ \cos(C+A) = \sin B.$$

3. If $A+B+C=\pi$, then $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}$ and hence we get that

$$\sin\left(\frac{A+B}{2}\right)=\sin\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cos\frac{C}{2}.\quad \text{Similarly,}$$

$$\sin\left(\frac{B+C}{2}\right)=\cos\frac{A}{2};\ \sin\frac{C+A}{2}=\cos\frac{B}{2};\ \cos\frac{A+B}{2}=\sin\frac{C}{2}$$

$$\cos\left(\frac{B+C}{2}\right)=\sin\frac{A}{2};\cos\left(\frac{C+A}{2}\right)=\sin\frac{B}{2}.$$

4. If $A+B+C=0$, then

$$\sin(A+B)=\sin(-C)=-\sin C.\ \text{Similarly, we get}$$

$$\sin(B+C)=-\sin A \quad \text{and} \quad \sin(C+A)=-\sin B$$

Again, $\cos(A+B)=\cos(-C)=\cos C$. Similarly,

$$\cos(B+C)=\cos A \quad \text{and} \quad \cos(C+A)=\cos B$$

We make use of all these identities in the following.

6.4.6 Solved Problems

1. Problem: *If* A, B, C *are the angles of a triangle, prove that*

(i) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$.

(ii) $\sin 2A+\sin 2B-\sin 2C=4\cos A\cos B\sin C$.

Solution

(i) $\sin 2A+\sin 2B+\sin 2C$

$$=2\sin(A+B)\cos(A-B)+2\sin C\cos C$$

$$=2\sin C\cos(A-B)+2\sin C\cos C\quad (\text{since } \sin(A+B)=\sin C)$$

$$=2\sin C\{\cos(A-B)+\cos C\}$$

$$=2\sin C\{\cos(A-B)-\cos(A+B)\}\quad (\text{since } \cos(A+B)=-\cos C)$$

$$=2\sin C(2\sin A\sin B)$$

$$=4\sin A\sin B\sin C.$$

(ii) $\sin 2A + \sin 2B - \sin 2C$

$= 2\sin(A+B)\cos(A-B) - 2\sin C\cos C$

$= 2\sin C\cos(A-B) - 2\sin C\cos C$

$= 2\sin C\{\cos(A-B) - \cos C\}$

$= 2\sin C\{\cos(A-B) + \cos(A+B)\}$

$= 2\sin C\{2\cos A\cos B\}$

$= 4\cos A\cos B\sin C.$

2. Problem : *If A, B, C are angles of a triangle, prove that*

(i) $\cos 2A + \cos 2B + \cos 2C = -4\cos A\cos B\cos C - 1.$

(ii) $\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A\sin B\cos C.$

Solution

(i) $\cos 2A + \cos 2B + \cos 2C$

$= 2\cos(A+B)\cos(A-B) + 2\cos^2 C - 1$

$= -2\cos C\cos(A-B) + 2\cos^2 C - 1$ (since $\cos(A+B) = -\cos C$)

$= -2\cos C\{\cos(A-B) - \cos C\} - 1$

$= -2\cos C\{\cos(A-B) + \cos(A+B)\} - 1$

$= -2\cos C(2\cos A\cos B) - 1 = -4\cos A\cos B\cos C - 1.$

(ii) $\cos 2A + \cos 2B - \cos 2C$

$= 2\cos(A+B)\cos(A-B) - (2\cos^2 C - 1)$

$= 1 - 2\cos C\cos(A-B) - 2\cos^2 C$

$= 1 - 2\cos C(\cos(A-B) + \cos C)$

$= 1 - 2\cos C\{\cos(A-B) - \cos(A+B)\}$

$= 1 - 2\cos C(2\sin A\sin B)$

$= 1 - 4\sin A\sin B\cos C.$

3. Problem : *If A, B, C are angles in a triangle, then prove that*

(i) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$

(ii) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$

Solution

(i) $\sin A + \sin B + \sin C$

$$= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$= 2\cos\frac{C}{2}\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\cos\frac{C}{2} \quad \left(\text{since } \sin\left(\frac{A+B}{2}\right) = \cos\frac{C}{2}\right)$$

$$= 2\cos\frac{C}{2}\left\{\cos\left(\frac{A-B}{2}\right) + \sin\frac{C}{2}\right\}$$

$$= 2\cos\frac{C}{2}\left\{\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)\right\} \quad \left(\text{since } \cos\left(\frac{A+B}{2}\right) = \sin\frac{C}{2}\right)$$

$$= 2\cos\frac{C}{2}\left(2\cos\frac{A}{2}\cos\frac{B}{2}\right)$$

$$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

(ii) $\cos A + \cos B + \cos C$

$$= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2}$$

$$= 1 + 2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) - 2\sin^2\frac{C}{2}$$

$$= 1 + 2\sin\frac{C}{2}\left\{\cos\left(\frac{A-B}{2}\right) - \sin\frac{C}{2}\right\}$$

$$= 1 + 2\sin\frac{C}{2}\left\{\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right\}$$

$$= 1 + 2\sin\frac{C}{2}\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right)$$

$$= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

4. Problem: *If* $A + B + C = \frac{\pi}{2}$, *then show that*

(i) $\sin^2 A + \sin^2 B + \sin^2 C = 1 - 2\sin A \sin B \sin C.$

(ii) $\sin 2A + \sin 2B + \sin 2C = 4\cos A \cos B \cos C.$

Solution

(i) $\sin^2 A + \sin^2 B + \sin^2 C$

$$= \frac{1}{2}\left\{2\sin^2 A + 2\sin^2 B + 2\sin^2 C\right\}$$

$$= \frac{1}{2}\{1 - \cos 2A + 1 - \cos 2B + 1 - \cos 2C\}$$

$$= \frac{1}{2}\left\{3 - \left(\cos 2A + \cos 2B + \cos 2C\right)\right\}$$

$$= \frac{1}{2}\left\{3 - \left(1 + 4\sin A \sin B \sin C\right)\right\} \text{ (from problem 3(ii) since } 2A + 2B + 2C = \pi \text{)}$$

$$= \frac{1}{2}\left(2 - 4\sin A \sin B \sin C\right)$$

$$= 1 - 2\sin A \sin B \sin C.$$

(ii) since $2A + 2B + 2C = \pi$, from problem 3(i), we get

$$\sin 2A + \sin 2B + \sin 2C = 4\cos A \cos B \cos C.$$

5. Problem: *If* $A + B + C = \frac{3\pi}{2}$, *prove that*

$$\cos 2A + \cos 2B + \cos 2C = 1 - 4\sin A \sin B \sin C.$$

Solution: $\cos 2A + \cos 2B + \cos 2C$

$$= 2\cos(A + B)\cos(A - B) + 1 - 2\sin^2 C$$

$$= 2(-\sin C)\cos(A - B) + 1 - 2\sin^2 C$$

$$\left(\text{since } \cos(A + B) = \cos\left(\frac{3\pi}{2} - C\right) = -\sin C\right)$$

$$= 1 - 2\sin C\left\{\cos(A - B) + \sin C\right\}$$

$$= 1 - 2\sin C\left\{\cos(A - B) - \cos(A + B)\right\} \quad \text{(as above)}$$

$$= 1 - 2\sin C\,(2\sin A \sin B) = 1 - 4\sin A \sin B \sin C.$$

6. Problem: *If* A, B, C *are angles of a triangle, then prove that*

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2} = 1 - 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

Solution: $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2}$

$$= \frac{1}{2}\left\{2\sin^2\frac{A}{2} + 2\sin^2\frac{B}{2} - 2\sin^2\frac{C}{2}\right\}$$

$$= \frac{1}{2}\{(1-\cos A) + (1-\cos B) - (1-\cos C)\}$$

$$= \frac{1}{2}\{1 - (\cos A + \cos B) + \cos C\}$$

$$= \frac{1}{2}\left\{1 - 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + 1 - 2\sin^2\frac{C}{2}\right\}$$

$$= \frac{1}{2}\left\{2 - 2\sin\frac{C}{2}\cos\frac{A-B}{2} - 2\sin^2\frac{C}{2}\right\} \left(\text{since } \cos\frac{A+B}{2} = \sin\frac{C}{2}\right)$$

$$= 1 - \sin\frac{C}{2}\left(\cos\frac{A-B}{2} + \sin\frac{C}{2}\right)$$

$$= 1 - \sin\frac{C}{2}\left(\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right) \quad \text{(as above)}$$

$$= 1 - \sin\frac{C}{2}\left(2\cos\frac{A}{2}\cos\frac{B}{2}\right) = 1 - 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

7. Problem: *If* A, B, C *are the angles in a triangle, then prove that*

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} = 1 + 4\sin\frac{\pi - A}{4}\sin\frac{\pi - B}{4}\sin\frac{\pi - C}{4}.$$

Solution: $\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}$

$$= 2\sin\left(\frac{A+B}{4}\right)\cos\left(\frac{A-B}{4}\right) + \cos\left(\frac{A+B}{2}\right) \quad \left(\text{since } \cos\frac{A+B}{2} = \sin\frac{C}{2}\right)$$

$$= 2\sin\left(\frac{A+B}{4}\right)\cos\left(\frac{A-B}{4}\right) + 1 - 2\sin^2\left(\frac{A+B}{4}\right)$$

$$= 1 + 2\sin\left(\frac{A+B}{4}\right)\left(\cos\left(\frac{A-B}{4}\right) - \sin\left(\frac{A+B}{4}\right)\right)$$

$$= 1 + 2\sin\left(\frac{\pi - C}{4}\right)\left(\cos\frac{A-B}{4} - \cos\left(\frac{\pi}{2} - \frac{A+B}{4}\right)\right)$$

$$= 1 + 2\sin\frac{\pi - C}{4}\left(-2\sin\frac{\pi - B}{4}\sin\left(\frac{A-\pi}{4}\right)\right)$$

$$= 1 + 4\sin\left(\frac{\pi - A}{4}\right)\sin\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - C}{4}\right).$$

8. Problem: *If* $A + B + C = 0$, *then prove that*

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 + 2\cos A \cos B \cos C.$$

Solution: $\cos^2 A + \cos^2 B + \cos^2 C$

$$= \frac{(1+\cos 2A)}{2} + \frac{(1+\cos 2B)}{2} + \frac{(1+\cos 2C)}{2}$$

$$= \frac{1}{2}\{3 + \cos 2A + \cos 2B + \cos 2C\}$$

$$= \frac{1}{2}\{3 + 2\cos(A+B)\cos(A-B) + 2\cos^2 C - 1\}$$

$$= \frac{1}{2}\{2 + 2\cos C \,.\, \cos(A-B) + 2\cos^2 C\}$$

(since $\cos(A+B) = \cos(-C) = \cos C$)

$$= 1 + \cos C\,(\cos(A-B) + \cos C)$$

$$= 1 + \cos C\,\{\cos(A-B) + \cos(A+B)\}$$

$$= 1 + \cos C\,(2\cos A \cos B)$$

$$= 1 + 2\cos A \cos B \cos C.$$

9. Problem: *If* $A + B + C = 2S$, *then prove that*

$$\cos(S-A) + \cos(S-B) + \cos(S-C) + \cos S = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Solution: $\cos(S-A) + \cos(S-B) + \cos(S-C) + \cos S$

$$= 2\cos\left(\frac{2S-A-B}{2}\right)\cos\left(\frac{B-A}{2}\right) + 2\cos\frac{2S-C}{2} \,.\, \cos\left(\frac{-C}{2}\right)$$

$$= 2\cos\frac{C}{2}\cos\left(\frac{B-A}{2}\right) + 2\cos\left(\frac{A+B}{2}\right)\cos\frac{C}{2} \quad \text{(since } 2S - C = A + B\text{)}$$

$$= 2\cos\frac{C}{2}\left\{\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)\right\}$$

$$= 2\cos\frac{C}{2}\left\{2\cos\frac{A}{2}\cos\frac{B}{2}\right\}$$

$$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Exercise 6(f)

I. **1.** If A, B, C are angles in a triangle, then prove that

(i) $\sin 2A - \sin 2B + \sin 2C = 4\cos A \sin B \cos C$.

(ii) $\cos 2A - \cos 2B + \cos 2C = 1 - 4\sin A \cos B \sin C$.

2. If A, B, C are angles in a triangle, then prove that

(i) $\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$.

(ii) $\cos A + \cos B - \cos C = -1 + 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$.

3. If A, B, C are angles in a triangle, then prove that

(i) $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A \sin B \cos C$.

(ii) $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2\sin A \sin B \cos C$.

4. If $A + B + C = \pi$, then prove that

(i) $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2\left(1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$.

(ii) $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} - \cos^2\frac{C}{2} = 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$.

5. In triangle ABC, prove that

(i) $\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = 4\cos\frac{\pi - A}{4}\cos\frac{\pi - B}{4}\cos\frac{\pi - C}{4}$

(ii) $\cos\frac{A}{2} + \cos\frac{B}{2} - \cos\frac{C}{2} = 4\cos\frac{\pi + A}{4}\cos\frac{\pi + B}{4}\cos\frac{\pi - C}{4}$

(iii) $\sin\frac{A}{2} + \sin\frac{B}{2} - \sin\frac{C}{2} = -1 + 4\cos\frac{\pi - A}{4}\cos\frac{\pi - B}{4}\sin\frac{\pi - C}{4}$

6. If $A + B + C = \frac{\pi}{2}$, then prove that $\cos 2A + \cos 2B + \cos 2C = 1 + 4\sin A \sin B \sin C$.

7. If $A + B + C = \frac{3\pi}{2}$, then prove that

(i) $\cos^2 A + \cos^2 B - \cos^2 C = -2\cos A \cos B \sin C$.

(ii) $\sin 2A + \sin 2B - \sin 2C = -4\sin A \sin B \cos C$.

8. If $A + B + C = 0$, then prove that

(i) $\sin 2A + \sin 2B + \sin 2C = -4\sin A \sin B \sin C$.

(ii) $\sin A + \sin B - \sin C = -4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$.

9. If $A + B + C + D = 2\pi$, then prove that

(i) $\sin A - \sin B + \sin C - \sin D = -4\cos\frac{A+B}{2}\sin\frac{A+C}{2}\cos\frac{A+D}{2}$.

(ii) $\cos 2A + \cos 2B + \cos 2C + \cos 2D = 4\cos(A+B)\cos(A+C)\cos(A+D)$.

10. If $A + B + C = 2S$, then prove that

(i) $\sin(S-A) + \sin(S-B) + \sin C = 4\cos\frac{S-A}{2}\cos\frac{S-B}{2}\sin\frac{C}{2}$.

(ii) $\cos(S-A) + \cos(S-B) + \cos C = -1 + 4\cos\frac{S-A}{2}\cos\frac{S-B}{2}\cos\frac{C}{2}$.

Key Concepts

- For any angle θ, $\cos^2\theta + \sin^2\theta = 1$.
- If $\cos\theta \neq 0$, then $1 + \tan^2\theta = \sec^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$.
- If $\sin\theta \neq 0$, then $1 + \cot^2\theta = \text{cosec}^2\theta$ or $\cot^2\theta = \text{cosec}^2\theta - 1$.
- The trigonometric ratios that are positive in different quadrants are given by

Quadrant	:	**I**	**II**	**III**	**IV**
Trigonometric ratio that is +ve	:	All	sine	tangent	cosine
Remember	:	All	Silver	Tea	Cups

- $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$; $\tan(-\theta) = -\tan\theta$.
- All trigonometric functions are periodic.

  The period of $f(x) = \sin x$ is 2π.

  The period of $f(x) = \cos x$ is 2π.

  The period of $f(x) = \tan x$ is π.
- If $y = f(x)$ is a periodic function with period *k*, then $g(x) = f(ax + b)$ is a periodic function with period $\frac{k}{|a|}$.
- If $y = f(x)$, $y = g(x)$ are periodic functions with *l, m* as the periods respectively, then for $a, b \in \mathbf{R}$, the function *h*, defined by

$$h(x) = af(x) + bg(x)$$

is a periodic function and the *l.c.m* of $\{l, m\}$ (if exists) is a period of *h*.

❖ (i) The range of $\sin x$ or $\cos x$ is $[-1, 1]$.
(ii) The range of $\tan x$ or $\cot x$ is **R**.
(iii) The range of $\sec x$ or $\operatorname{cosec} x$ is $(-\infty, -1] \cup [1, \infty)$.

❖ For any two angles A, B

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

❖ If none of A, B, A + B, A – B is an odd multiple of $\frac{\pi}{2}$, then

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

❖ If none of A, B, A + B, A – B is an integral multiple of π, then

$$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}.$$
$$\cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}.$$

❖ If A, B, C $\in$ **R**

$$\sin(A+B+C) = \sum(\sin A \cos B \cos C) - \sin A \sin B \sin C.$$
$$\cos(A+B+C) = \cos A \cos B \cos C - \sum(\cos A \sin B \sin C).$$

❖ If none of A, B, A + B + C is an odd multiple of $\frac{\pi}{2}$ and at least one of A + B, B + C, C + A is not an odd multiple of $\frac{\pi}{2}$, then

$$\tan(A+B+C) = \frac{\sum(\tan A) - \Pi \tan A}{1 - \sum(\tan A \tan B)}.$$

❖ If none of A, B, C, A + B + C is an integral multiple of π and at least one of B + C, C + A, A + B is not an integral multiple of π, then

$$\cot(A+B+C) = \frac{\sum(\cot A) - \Pi(\cot A)}{1 - \sum(\cot A \cot B)}.$$

- For any $A \in \mathbf{R}$,

(i) $\sin 2A = 2\sin A \cos A = \dfrac{2\tan A}{1+\tan^2 A}$.

(ii) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$

$= 1 - 2\sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$.

(iii) If A and $2A$ are not odd multiples of $\dfrac{\pi}{2}$, then $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$.

(iv) If $2A$ is not an integral multiple of π, then $\cot 2A = \dfrac{\cot^2 A - 1}{2\cot A}$.

- For any $A \in \mathbf{R}$,

(i) $\sin 3A = 3\sin A - 4\sin^3 A$.

(ii) $\cos 3A = 4\cos^3 A - 3\cos A$.

(iii) If $3A$ is not an odd multiple of $\dfrac{\pi}{2}$, then $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$.

(iv) If $3A$ is not an integral multiple of π, then $\cot 3A = \dfrac{3\cot A - \cot^3 A}{1-3\cot^2 A}$.

- For any $A \in \mathbf{R}$,

(i) $\sin A = \pm\sqrt{\dfrac{1-\cos 2A}{2}}$.

(ii) $\cos A = \pm\sqrt{\dfrac{1+\cos 2A}{2}}$.

(iii) If A is not an odd multiple of $\dfrac{\pi}{2}$, then $\tan A = \pm\sqrt{\dfrac{1-\cos 2A}{1+\cos 2A}}$.

- For any $A \in \mathbf{R}$,

(i) $\sin\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{2}}$ (ii) $\cos\dfrac{A}{2} = \pm\sqrt{\dfrac{1+\cos A}{2}}$.

(iii) If A is not an odd multiple of π, then $\tan\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{1+\cos A}}$.

- Transformations from product to sums are

(i) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(ii) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(iii) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(iv) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

❖ Transformations from sums to products are

(i) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right).$

(ii) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right).$

(iii) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right).$

(iv) $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right).$

❖ If $A + B + C = \pi$, then

(i) $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C\cdot$

(ii) $\sin 2A + \sin 2B - \sin 2C = 4\cos A\cos B\sin C\cdot$

(iii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C\cdot$

(iv) $\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A\sin B\cos C\cdot$

(v) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$

(vi) $\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.$

(vii) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$

(viii) $\cos A + \cos B - \cos C = -1 + 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$

(ix) $\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} = 1 + 4\sin\frac{\pi-A}{4}.\sin\frac{\pi-B}{4}.\sin\frac{\pi-C}{4}.$

(x) $\sin\frac{A}{2} + \sin\frac{B}{2} - \sin\frac{C}{2} = 1 + 4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\sin\frac{\pi-C}{4}.$

(xi) $\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = 4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}.$

(xii) $\cos\frac{A}{2} + \cos\frac{B}{2} - \cos\frac{C}{2} = 4\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi-C}{4}.$

Historical Note

In early Indian mathematics, trigonometry formed an integral part of Astronomy. References to trigonometric concepts are found in 'Surya Siddhanta', *Varahamihira's* (505 - 587) 'Pancha Siddantika' and *Brahmagupta's* (628 A.D.) 'Brahmasphuta Siddhanta'. A detailed and systematic study on the subject was made by '*Bhaskaracharya*' (12th century A.D.) in his 'Siddhanta Siromani'.

Aryabhatta and *Varahamihira* were great astronomers and mathematicians of his times in India. Varahamihira's 'Pancha Siddhantika' is a monumental work in astronomy. It gives the description of the five siddhantas namely, Paulisa, Romaka, Vasista, Soura and Paitamaha of earlier period. His knowledge of trigonometry was amazing. Several of his formulas are the standard results that we find in modern trigonometry. The technical terms he used for various trigonometric functions are very interesting and highly meaningful. His magnum opus is the well known "Brihat Samhita". Besides being a mathematician - astronomer he was equally a great hydrologist and was an expert in locating the ground water deposits.

Answers

Exercise 6(a)

I. 1. **(i)** $\tan\theta$ **(ii)** $\tan\theta$ **(iii)** $-\operatorname{cosec}\theta$ **(iv)** $\sec\theta$

2. **(i)** $-\frac{1}{\sqrt{2}}$ **(ii)** 0 **(iii)** 2 **(iv)** 1

3. **(i)** 2 (ii) $\frac{1}{2}$ **(iii)** 0 **(iv)** (a) $\sqrt{2}$, (b) $\frac{(\sqrt{3}+1)}{2}$

4. **(i)** $\frac{2\sqrt{2}}{3}$ and $-2\sqrt{2}$ **(ii)** (a) $-\sqrt{1-t^2}$ (b) $\frac{-\sqrt{1-t^2}}{t}$

(iii) 1 **(iv)** $\frac{-4}{5}$, 2nd quadrant

5. **(i)** 2 **(ii)** 1st quadrant and $\frac{12}{13}$

II. 2. **(i)** $\frac{2}{\sqrt{3}}$ **(iii)** $\frac{2a^2-b^2}{b^2}$ **3.** (i) $\frac{2}{3}$

III. 2. **(iv)** x

3. **(i)** $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1$ **(ii)** $\left(\frac{x}{a}\right)^{1/2}+\left(\frac{y}{b}\right)^{1/2}=1$

(iii) $xy = ab$; **(iv)** $(x^2y)^{2/3}-(xy^2)^{2/3}=1$

Exercise 6(b)

I. **1.** $\frac{2\pi}{3}$ **2.** $\frac{\pi}{5}$ **3.** $\frac{5\pi}{2}$

4. π **5.** $\frac{6\pi}{n(n+1)(2n+1)}$ **6.** $\sin 3\pi x$

7. $\cos\frac{2\pi}{7}x$

Exercise 6(c)

I. **1.** (i) $\frac{1}{2}$ (ii) 0 (iii) 1 (iv) 4

(v) $\frac{1}{2}$

2. (i) $\sin 85^0$ (ii) $\sqrt{2}\cos\left(\theta+\frac{\pi}{4}\right)$ (iii) $\frac{1-\tan\alpha}{1+\tan\alpha}$

3. (i) 236^0 (ii) $\frac{16}{65}$ (iii) $\sqrt{3}$

(iv) 1 (v) 0 (vi) 0

5. (i) $\frac{3+\sqrt{3}}{4\sqrt{2}}$ (ii) $\frac{-(\sqrt{3}+1)}{4\sqrt{2}}$ (iii) $\frac{1}{\sqrt{2}}\sin A$ (iv) $\frac{3-\sqrt{3}}{4\sqrt{2}}$

6. (i) -5 and $+5$ (ii) $-\sqrt{2}$ and $+\sqrt{2}$

7. (i) $[-20, 30]$ (ii) $[-18, 10]$

II. **1.** (i) $\frac{-3}{4}$ and $\frac{3}{5}$ (ii) $\frac{3}{4}$ (iii) $\frac{4}{5}$

2. (i) sin A cos B cos C + cos A sin B cos C – cos A cos B sin C + sinA sin B sin C

(ii) cosA cosB cosC + sinA sinB cos C + sinA cos B sinC – cos A sinB sinC

Exercise 6(d)

I. **1.** (i) $\tan\theta$ (ii) $\cot^3\theta$

2. (i) $\sqrt{3}$ (ii) $-\frac{\sqrt{5}}{4}$ (iii) $\frac{(\sqrt{5}+1)}{8}$

3. (i) $(8\cos^3\theta - 4\cos\theta)$ (ii) $1-\frac{3}{4}\sin^2 2A$ (iii) $\tan(\theta/2)$

4. (i) $\frac{44}{125}, -\frac{24}{7}$ (ii) $\frac{-4}{3}$

5. (i) -1 and 2 (ii) 3 and 5

6. $a = 1$ and $b = 11$

7. (i) π (ii) 24 (iii) π

(iv) π (v) 2π

II. **1.** (i) $\frac{24}{25}, \frac{7}{25}$ (ii) $0 < A < \pi/6$ or $\pi/3 < A < \pi/2$

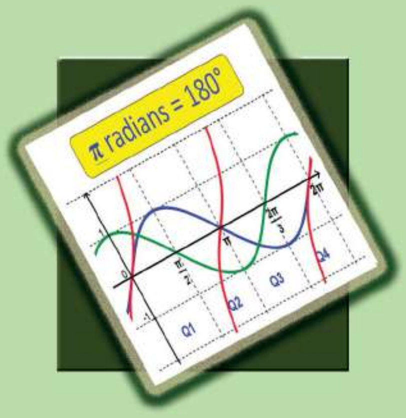
2. (i) $2+\sqrt{3}$ (ii) $\frac{\sqrt{5}}{2}, -2$ (iii) $\frac{-3}{\sqrt{13}}, \frac{-2}{\sqrt{13}}$

4. (i) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (ii) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (iii) $(2-\sqrt{3}), (\sqrt{2}-1)$

6. (ii) $\frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}}$ (iii) $\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}$

Exercise 6 (e)

III. **1.** 0



Chapter 7

Trigonometric Equations

"Brahmagupta was the first Indian writer, so far as we know, who applied algebra to astronomy to a great extent"

– D.E. Smith

Brahmagupta

(598–668)

***Brahmagupta** was born in Bhinmal city in the state of Rajasthan. Brahmasphuta Siddhanta is his most famous work. He lived and worked in the great astro nomical centre of ancient India - Ujjain. He made significant contributions to Trigonometry.*

Introduction

In earlier classes, we have solved equations $f(x) = 0$ such as $ax + b = 0 \ (a \neq 0)$, $ax^2 + bx + c = 0$ $(a \neq 0)$ etc. A solution for the equation $f(x) = 0$ means a number x_0 that satisfies the given equation. That is, $f(x_0) = 0$. In this chapter, we solve equations involving the trigonometric functions like

$$4 \sin^2 x - 3 \sin x - 1 = 0$$

$$3 \tan^2 x + 4 \tan x - 7 = 0$$

7.1 General solutions of trigonometric equations

In this section we derive general solution of simple trigonometric equations like $\sin x = k$, $\cos x = k$ etc.

7.1.1 Definition

An equation consisting of the trigonometric functions of a variable angle $\theta \in \mathbf{R}$ is called a ***'trigonometric equation'.***

7.1.2 Definition

The values of the variable angle θ, that is any number θ, satisfying the given trigonometric equation is called a ***'solution'*** *of the equation. The set of all solutions of a trigonometric equation is called the* ***'solution set'*** *of the equation. A* ***'general solution'*** *of the equation is an expression of the form $\theta_0 + f(n)$ where θ_0 is a particular solution and $f(n)$ is a function of $n \in \mathbf{Z}$ involving π.*

7.1.3 Example

The equation $\sin\theta = \frac{1}{2}$ has a solution $\theta = \frac{\pi}{6}$. But $\theta = \frac{5\pi}{6}, \frac{13\pi}{6}$... are also solutions of this equation. The general solution is $\theta = 2n\pi + \frac{\pi}{6}$ or $2n\pi + \frac{5\pi}{6}$ $(n \in \mathbf{Z})$. If θ is a solution of a trigonometric equation, then $2n\pi + \theta$ $(n \in \mathbf{Z})$ is also a solution of the same equation since 2π is a period of all trigonometric functions.

Now we define the concept of the principal solution and give formula (or method) to find general solution of trigonometric equations.

7.1.4 Definition

(a) The function $f(x) = \sin x$ has domain $\mathbf{R}$ and range $[-1, 1]$. But if we define the function $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ by $f(x) = \sin x$, then f is a bijection. Hence, for each $k \in [-1, 1]$, there exists unique $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin\theta = k$. This $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is called the ***'principal solution'*** *of the equation $\sin x = k$. If $k \in \mathbf{R} \setminus [-1, 1]$, then the equation $\sin x = k$ has no solution.*

For example, the equation $\sin x = \frac{\sqrt{3}}{2}$ has principal solution $x = \frac{\pi}{3}$ and the equation $\sin x = \frac{-1}{2}$ has principal solution $x = \frac{-\pi}{6}$ whereas the equation $\sin x = \sqrt{2}$ has no solution.

Now we give the definitions of principal solutions of other trigonometric functions in the following.

(b) *Cosine function is a bijection from* $[0, \pi]$ *onto* $[-1, 1]$. *Hence for* $k \in [-1, 1]$, *there exists unique* $\alpha \in [0, \pi]$ *such that* $\cos \alpha = k$. *This '*α*' is called the* ***'principal solution'*** *of the equation* $\cos x = k$.

(c) *The tangent function is a bijection from* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *onto* $\mathbf{R}$ *so that, for any* $k \in \mathbf{R}$, *the unique* $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *such that* $\tan \alpha = k$ *is the* ***'principal solution'*** *of the equation* $\tan x = k$.

(d) *If* $k \neq 0$, $\cot x = k$ *if and only if* $\tan x = \frac{1}{k}$ *and in this case* $\tan x = \frac{1}{k}$ *has a solution. Therefore, when* $k \neq 0$, *these two equations* ($\cot x = k$; $\tan x = \frac{1}{k}$) *have same solution set. But the* ***principal solution*** *of the equation* $\cot x = k$ *is the unique real number* $\alpha \in (0, \pi)$ *such that* $\cot \alpha = k$.

(e) *Sec* $x = k$ *iff* $\cos x = \frac{1}{k}$. *The second equation has a solution if and only if* $|k| \geq 1$. *In this case, the solution set of* $\sec x = k$ *is the same as the solution set of* $\cos x = \frac{1}{k}$. *The principal solution of* $\cos x = \frac{1}{k}$ *may be referred to as the* ***principal solution*** *of* $\sec x = k$.

(f) *Cosec* $x = k$ *if and only if* $\sin x = \frac{1}{k}$. *The second equation has a solution if and only if* $|k| \geq 1$. *The solution set of* $\operatorname{cosec} x = k$ *is the same as the solution set of* $\sin x = \frac{1}{k}$. *The principal solution of* $\sin x = \frac{1}{k}$ *may be referred to as the* ***principal solution*** *of* $\operatorname{cosec} x = k$.

7.1.5 General solution of the equations $\sin x = 0$, $\cos x = 0$ and $\tan x = 0$

(i) If $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $\sin\theta = 0$ if and only if $\theta = 0$. Hence the principal solution of $\sin x = 0$ is 0.

Let θ be any real number such that $\sin\theta = 0$. Then there exists an integer k such that

$$2k\pi \le \theta < 2(k+1)\pi \quad \left(\text{Here } k = \left[\frac{\theta}{2\pi}\right], \text{ the integral part of } \frac{\theta}{2\pi}\right)$$

That is, $0 \le \theta - 2k\pi < 2\pi$.

Since θ and $\theta - 2k\pi$ are coterminal angles, we get

$0 = \sin\theta = \sin(\theta - 2k\pi)$.

Hence we get $\theta - 2k\pi = 0$ or $\theta - 2k\pi = \pi$.

That is, $\theta = 2k\pi$ or $(2k+1)\pi$.

Thus we get $\sin\theta = 0$ if and only if $\theta = n\pi$ for some integer n. Hence the **general solution** of the equation $\sin x = 0$ is $x = n\pi + 0 = n\pi,\ n \in \mathbf{Z}$.

(ii) Clearly, the principal solution of the equation $\cos x = 0$ is $x = \frac{\pi}{2}$. Now,

$$\cos x = 0 \Leftrightarrow \sin\left(x - \frac{\pi}{2}\right) = 0 \quad \Leftrightarrow \quad x - \frac{\pi}{2} = n\pi,\ n \in \mathbf{Z}$$

$$\Leftrightarrow x = n\pi + \frac{\pi}{2}, n \in \mathbf{Z} \quad \Leftrightarrow \quad x = (2n+1)\frac{\pi}{2}, n \in \mathbf{Z}.$$

Therefore, $x = (2n+1)\frac{\pi}{2}, n \in \mathbf{Z}$ is the **general solution** of $\cos x = 0$.

(iii) Clearly, the principal solution of $\tan x = 0$ is $x = 0$.

$$\tan x = 0 \Leftrightarrow \sin x = 0 \Leftrightarrow x = n\pi,\ n \in \mathbf{Z}.$$

Now, Therefore, the **general solution** of the equation $\tan x = 0$ is $x = n\pi,\ n \in \mathbf{Z}$.

(iv) We know that $x = \frac{\pi}{2}$ is the principal solution of the equation $\cot x = 0$. Now, for any $x \in \mathbf{R}$,

$$\cot x = 0 \Leftrightarrow \cos x = 0 \Leftrightarrow x = (2n+1)\frac{\pi}{2},\ n \in \mathbf{Z}.$$

Therefore, the **general solution** of the equation $\cot x = 0$ is $x = (2n+1)\frac{\pi}{2},\ n \in \mathbf{Z}$.

7.1.6 General solution of the equation $\sin x = k$ $(-1 \le k \le 1)$

Let $k \in [-1, 1]$ and α be the principal solution of the equation $\sin x = k$.

(That is, $\alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and $\sin\alpha = k$). Let θ be any solution of $\sin x = k$.

Now

$$\sin\theta = k = \sin\alpha \iff \sin\theta - \sin\alpha = 0$$

$$\iff 2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\iff \cos\frac{\theta+\alpha}{2} = 0 \text{ or } \sin\left(\frac{\theta-\alpha}{2}\right) = 0.$$

From 7.1.5 (ii) above, we get that

$$\cos\left(\frac{\theta+\alpha}{2}\right) = 0 \iff \frac{\theta+\alpha}{2} = (2n+1)\frac{\pi}{2},\ n \in \mathbf{Z}$$

$$\iff \theta = (2n+1)\pi - \alpha,\ n \in \mathbf{Z}$$

Now, from 7.1.5 (i),

$$\sin\left(\frac{\theta-\alpha}{2}\right) = 0 \iff \frac{\theta-\alpha}{2} = n\pi,\ n \in \mathbf{Z} \iff \theta = 2n\pi + \alpha,\ n \in \mathbf{Z}.$$

Thus $\theta = n\pi + (-1)^n\alpha$, where $n \in \mathbf{Z}$.

$\therefore$ General solution of the equation $\sin x = k\ (-1 \le k \le 1)$ is

$$x = n\pi + (-1)^n\alpha,\ n \in \mathbf{Z},$$

where α is the principal (or any) solution of the equation.

Thus the solution set of the equation $\sin x = k$ is $\{n\pi + (-1)^n\alpha \mid n \in \mathbf{Z}\}$.

7.1.7 General solution of $\cos x = k\ (-1 \le k \le 1)$

Let $k \in [-1, 1]$ and α be the principal solution of the equation $\cos x = k$. (That is, $\alpha \in [0, \pi]$). If θ is any solution of the equation $\cos x = k$, then

$$\cos\theta = k = \cos\alpha$$

$$\iff \cos\theta - \cos\alpha = 0$$

$$\iff 2\sin\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\iff \sin\frac{\theta+\alpha}{2} = 0 \quad\text{or}\quad \sin\frac{\theta-\alpha}{2} = 0$$

Now $\sin\frac{\theta+\alpha}{2} = 0 \iff \frac{\theta+\alpha}{2} = n\pi,\ n \in \mathbf{Z}$ by 7.1.5 (i)

$$\iff \theta = 2n\pi - \alpha.$$

Again, $\sin\frac{\theta-\alpha}{2} = 0 \iff \frac{\theta-\alpha}{2} = n\pi,\ n \in \mathbf{Z}$ as above

$$\iff \theta = 2n\pi + \alpha,\ n \in \mathbf{Z}$$

Hence, $\theta = 2n\pi \pm \alpha$ where $n \in \mathbf{Z}$.

Therefore, general solution of the equation $\cos x = k\ (-1 \le k \le 1)$ is $x = 2n\pi \pm \alpha,\ n \in \mathbf{Z}$.

where α is the principal (or any) solution.

Thus, the solution set of the equation $\cos x = k$ is $\{2n\pi \pm \alpha \mid n \in \mathbf{Z}\}$, where α is the principal solution.

7.1.8 General solution of the equation $\tan x = k$ ($k \in \mathbf{R}$)

Let $k \in \mathbf{R}$ and α be the principal solution of $\tan x = k$. $\left(\text{observe that } \alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\right)$. If θ is any solution of $\tan x = k$, then

$$\tan\theta = k = \tan\alpha \Leftrightarrow \frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha} \Leftrightarrow \sin\theta\cos\alpha - \cos\theta\sin\alpha = 0.$$

$$\Leftrightarrow \sin(\theta - \alpha) = 0 \Leftrightarrow \theta - \alpha = n\pi \text{ where } n \in \mathbf{Z} \text{ from 7.1.5(i)}$$

$$\Leftrightarrow \theta = n\pi + \alpha \quad \text{where } n \in \mathbf{Z}$$

Hence, the general solution of the equation $\tan x = k$ ($k \in \mathbf{R}$) is $x = n\pi + \alpha$, $n \in \mathbf{Z}$, where α is the principal (or any) solution of $\tan x = k$.

Thus the solution set of the equation $\tan x = k$ is $\{n\pi + \alpha \mid n \in \mathbf{Z}\}$, where α is the principal solution of $\tan x = k$.

7.1.9 General solution of the equation $\sec x = k$ $(k \in (-\infty, -1] \cup [1, \infty))$

As mentioned in 7.1.4 (*e*), the solution set of $\sec x = k$ is nonempty only when $|k| \geq 1$. Now let $k \in (-\infty, -1] \cup [1, \infty)$ and α be the principal solution of $\cos x = \frac{1}{k}$ so that $\alpha \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and $\cos\alpha = \frac{1}{k}$ and hence $\sec\alpha = k$. Now, for any $x \in \mathbf{R}$,

$$\sec x = k \Leftrightarrow \cos x = \frac{1}{k} \Leftrightarrow x = 2n\pi \pm \alpha \quad \text{(by 7.1.7)}$$

Thus the general solution of the equation $\sec x = k$ is $2n\pi \pm \alpha$, $n \in \mathbf{Z}$, where α is the principal (or any) solution of $\cos x = \frac{1}{k}$.

7.1.10 General solution of the equation $\operatorname{cosec} x = k$ $(k \in (-\infty, -1] \cup [1, \infty))$

As mentioned in 7.1.4 (*f*), the solution set is nonempty only when $|k| \geq 1$. Now let $k \in (-\infty, -1] \cup [1, \infty)$ and α be the principal solution of $\sin x = \frac{1}{k}$. That is, $\alpha \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ and $\sin\alpha = \frac{1}{k}$. Then $\operatorname{cosec}\alpha = k$ (note that $k \neq 0$). Now, for any $x \in \mathbf{R}$,

$$\operatorname{cosec} x = k \iff \sin x = \frac{1}{k} \iff x = n\pi + (-1)^n \alpha,\ n \in \mathbf{Z} \quad \text{(by 7.1.6)}$$

Thus the general solution of the equation $\operatorname{cosec} x = k$ is $n\pi + (-1)^n \alpha,\ n \in \mathbf{Z}$,

where α is the principal (or any) solution of $\sin x = \frac{1}{k}$.

7.1.11 General solution of the equation $\cot x = k$ ($k \in \mathbf{R}$)

As in 7.1.4 (*d*) the equation $\cot x = k$ has a solution for all $k \in \mathbf{R}$.

Case (i) : Let $k \in \mathbf{R} \setminus \{0\}$ and α be the principal solution of $\cot x = k$.

So that $\alpha \in (0, \pi)$ and $\cot \alpha = k$. Now, for any $x \in \mathbf{R}$,

$$\cot x = k \iff \tan x = \frac{1}{k} \iff x = n\pi + \alpha,\ n \in \mathbf{Z}.$$

Case (ii) : Let $k = 0$ and α be the principal solution of $\cos x = 0$ (i.e. $\cot x = 0$). Then $\alpha \in (0, \pi)$ and $\cos \alpha = 0$. Hence $\alpha = \frac{\pi}{2}$. Now, for any $x \in \mathbf{R}$,

$$\cot x = 0 \iff \cos x = 0 \iff x = (2n+1)\frac{\pi}{2},\ n \in \mathbf{Z}$$

$$\iff x = n\pi + \frac{\pi}{2} = n\pi + \alpha,\ n \in \mathbf{Z}.$$

Thus, in either case, the general solution of the equation $\cot x = k$ is $n\pi + \alpha, n \in \mathbf{Z}$, where α is the principal (or any) solution of $\cot x = k$.

7.1.12 Note

1. The solution sets of the trigonometric equations discussed above remain unchanged even if we take any solution in place of α instead of the principal solution.

2. The principal solutions and general solutions of the trigonometric equations given above are summarized in table 7.1.

Table 7.1

Serial No.	The equation $f(x) = k$	Range of k	The interval in which the principal solution a lies	General Solution
1.	$\sin x = k$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$n\pi + (-1)^n \alpha, n \in \mathbf{Z}$
2.	$\cos x = k$	$[-1, 1]$	$[0, \pi]$	$2n\pi \pm \alpha, n \in \mathbf{Z}$
3.	$\tan x = k$	$\mathbf{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$n\pi + \alpha, n \in \mathbf{Z}$
4.	$\operatorname{cosec} x = k$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$	$n\pi + (-1)^n \alpha, n \in \mathbf{Z}$
5.	$\sec x = k$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$	$2n\pi \pm \alpha, n \in \mathbf{Z}$
6.	$\cot x = k$	$\mathbf{R}$	$(0, \pi)$	$n\pi + \alpha, n \in \mathbf{Z}$

7.1.13 General solution of the equation $\sin^2 x = k$ ($k \in [0, 1]$)

The equation $\sin^2 x = k$ has a solution if and only if $k \in [0, 1]$. In this case there exists $\alpha \in \mathbf{R}$ such that $\sin^2 \alpha = k$. Now

Method (i) : $\sin^2 x = k \Leftrightarrow \sin^2 x = \sin^2 \alpha \Leftrightarrow 1 - 2\sin^2 x = 1 - 2\sin^2 \alpha$

$$\Leftrightarrow \cos 2x = \cos 2\alpha$$

$$\Leftrightarrow 2x = 2n\pi \pm 2\alpha, n \in \mathbf{Z}$$

$$\Leftrightarrow x = n\pi \pm \alpha, n \in \mathbf{Z}.$$

Thus, the general solution of the equation $\sin^2 x = k$ $(k \in [0, 1])$ is $n\pi \pm \alpha$, $n \in \mathbf{Z}$ (where α is a solution of $\sin^2 x = k$).

Method(ii) : $\sin^2 x = k = \sin^2 \alpha \Leftrightarrow \sin x = \sin \alpha$

or $\sin x = -\sin \alpha = \sin(-\alpha)$ so that the solution set of $\sin^2 x = k$ is the union of the solution sets of $\sin x = \sin \alpha$ and $\sin x = \sin(-\alpha)$. The general solution of $\sin x = \sin \alpha$ is $n\pi + (-1)^n \alpha$, $n \in \mathbf{Z}$ and the general solution of $\sin x = \sin(-\alpha)$ is $n\pi + (-1)^n(-\alpha) = n\pi + (-1)^{n+1}\alpha$, $n \in \mathbf{Z}$.

Thus the general solution of $\sin^2 x = k$ is $n\pi \pm \alpha$ (where α is a solution).

7.1.14 Note

As above we can prove that the general solutions of the equations

(i) $\cos^2 x = k$ is $n\pi \pm \alpha$, $n \in \mathbf{Z}$ if $k \in [0, 1]$ and $\cos^2 \alpha = k$.

(ii) $\tan^2 x = k$ is $n\pi \pm \alpha$, $n \in \mathbf{Z}$ if $k \in [0, \infty)$ and $\tan^2 \alpha = k$.

7.2 Simple trigonometric equations - solutions

In this section we solve a more general trigonometric equation.

7.2.1 General solution of the equation $a \cos x + b \sin x = c$ when $a, b, c \in \mathbb{R}$ such that $a^2 + b^2 \neq 0$ and $a + c \neq 0$.

Method (i): Given equation is $a \cos x + b \sin x = c$

$$\Rightarrow \quad a\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) + b\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) = c \quad (\because a+c \neq 0,\ x \neq (2n+1)\pi)$$

$$\Rightarrow \quad a\left(1-\tan^2\frac{x}{2}\right) + 2b\tan\frac{x}{2} = c\left(1+\tan^2\frac{x}{2}\right)$$

$$\Rightarrow \quad (a+c)\tan^2\frac{x}{2} - 2b\tan\frac{x}{2} + (c-a) = 0.$$

This is a quadratic equation in $\tan\frac{x}{2}$, so that

$$\tan\frac{x}{2} = \frac{2b \pm \sqrt{4b^2 - 4\left(c^2 - a^2\right)}}{2\,(a+c)}$$

$$= \frac{b \pm \sqrt{b^2 - c^2 + a^2}}{(a+c)}$$

$\therefore$ The given equation has solution if and only if $c^2 \leq a^2 + b^2$

$$\text{i.e., } c \in \left[-\sqrt{a^2+b^2},\ \sqrt{a^2+b^2}\right]$$

If $c \in \left[-\sqrt{a^2+b^2},\ \sqrt{a^2+b^2}\right]$, then we solve $\tan\frac{x}{2} = k$ where

$$k = \frac{b \pm \sqrt{b^2 + a^2 - c^2}}{a+c}.$$

Let $k_1 = \dfrac{b + \sqrt{b^2 + a^2 - c^2}}{a+c}$, $k_2 = \dfrac{b - \sqrt{b^2 + a^2 - c^2}}{a+c}$.

If α_i is the principal solution of $\tan\frac{x}{2} = k_i$, α_i lies in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

and the general solution is given by

$$\frac{x}{2} = n\pi + \alpha_i,\ n \in \mathbf{Z}\quad (i = 1, 2)$$

$$\text{or}\quad x = 2n\pi + 2\alpha_i,\ n \in \mathbf{Z}\ (i = 1, 2).$$

Method (ii): As observed above, the given equation has a real solution if and only if

$c \in \left[-\sqrt{a^2 + b^2}, \sqrt{a^2 + b^2}\right]$.

That is $|c| \le \sqrt{a^2 + b^2}$ or $|c| \le r$ where $r = \sqrt{a^2 + b^2} \Rightarrow \frac{|c|}{r} \le 1$.

Choose β such that $\cos\beta = \frac{a}{r}$ and $\sin\beta = \frac{b}{r}$ (If $a \ne 0$, β is chosen such that $\tan\beta = \frac{b}{a}$ since the range of $\tan x$ is **R**. If $a = 0$ then $b = \pm r$. Hence we take β to be $\pm\frac{\pi}{2}$ accordingly). Then

$a\cos x + b\sin x = c$

$$\Rightarrow r(\cos\beta\cos x + \sin\beta\sin x) = c$$

$$\Rightarrow \cos(x - \beta) = \frac{c}{r} \text{ and } \left|\frac{c}{r}\right| \le 1.$$

Hence there exists $\alpha \in [0, \pi]$ such that $\cos\alpha = \frac{c}{r}$. That is, $\cos(x - \beta) = \cos\alpha$. Thus $x = \alpha + \beta$ is the principal solution. Hence $x = 2n\pi \pm \alpha + \beta,\ n \in \mathbf{Z}$ is the general solution.

7.2.2 Note: The above equation can also be solved by choosing a $\phi \in \mathbf{R}$ such that $a = r\sin\phi$ and $b = r\cos\phi$. But it is same as second method in which $\theta = \frac{\pi}{2} - \phi$.

7.2.3 Solved Problems

1. Problem: *Solve* $\sin x = \frac{1}{\sqrt{2}}$.

Solution

Method 1: $\sin x = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$ and $\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus by 7.1.6, $x = \frac{\pi}{4}$ is the principal solution and $x = n\pi + (-1)^n\frac{\pi}{4},\ n \in \mathbf{Z}$ is the general solution. Therefore, the solution

$$\text{set} = \left\{\ldots, \frac{-7\pi}{4}, \frac{-5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4} \ldots\right\}.$$

Method 2: Put $a = 0,\ b = 1,\ c = \frac{1}{\sqrt{2}}$ 7.2.1 (Method (ii)). Then $\beta = \frac{\pi}{2}$ and $\alpha = \frac{\pi}{4}$. Hence $x = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{2},\ n \in \mathbf{Z}$. It can be observed that here also the solution set

$$= \left\{ \ldots, \frac{-7\pi}{4}, \frac{-5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4} \ldots \right\}.$$

2. Problem: *Solve* $\sin 2\theta = \frac{\sqrt{5}-1}{4}$.

Solution: $\sin 2\theta = \frac{\sqrt{5}-1}{4} = \sin\frac{\pi}{10}$ and $\frac{\pi}{10} \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$.

$\Rightarrow 2\theta = \frac{\pi}{10}$ and hence $\frac{\pi}{20}$ is the principal solution.

General solution is given by $2\theta = n\pi + (-1)^n \frac{\pi}{10},\quad n \in \mathbf{Z}$ or

$$\theta = \frac{n}{2}\pi + (-1)^n \frac{\pi}{20},\quad n \in \mathbf{Z}.$$

Note: It is important to note that in the above problem we obtained the principal solution as $2\theta = \frac{\pi}{10}$ or $\theta = \frac{\pi}{20}$. Then we get general solution for 2θ only (not for θ) since we are solving the equation $\sin 2\theta = k$. That means general solution is $2\theta = n\pi + (-1)^n \frac{\pi}{10},\ n \in \mathbf{Z}$ but not $\theta = n\pi + (-1)^n \frac{\pi}{20}$.

3. Problem: *Solve* $\tan^2\theta = 3$.

Solution: $\tan^2\theta = 3 \Rightarrow \tan\theta = \pm\sqrt{3} = \tan\left(\pm\frac{\pi}{3}\right)$ and $\pm\frac{\pi}{3} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$\therefore \alpha = \pm\frac{\pi}{3}$ are the principal solutions of the given equation. General solution is given by

$n\pi \pm \frac{\pi}{3},\ n \in \mathbf{Z}$.

4. Problem: *Solve* $3\operatorname{cosec} x = 4 \sin x$.

Solution: Given that $4\sin^2 x = 3 \Rightarrow \sin x = \pm\frac{\sqrt{3}}{2}$.

$\therefore$ Principal solutions are $x = \pm\frac{\pi}{3}$.

General solution is given by $x = n\pi \pm \frac{\pi}{3},\ n \in \mathbf{Z}$.

5. Problem: *If* x *is acute and* $\sin\left(x+10^0\right)=\cos\left(3x-68^0\right)$, *find* x in degrees.

Solution: Given that

$$\sin\left(x+10^0\right)=\cos\left(3x-68^0\right)=\sin\left(90^0+3x-68^0\right)=\sin\left(22^0+3x\right)$$

$$\therefore\ x+10^0=n\left(180^0\right)+(-1)^n\left(22^0+3x\right)$$

If $n=2k$, then $x+10^0=2k\left(180^0\right)+22^0+3x$

$$\Rightarrow\ 2x=-k\left(360^0\right)-12^0\Rightarrow x=\frac{-k\left(360^0\right)-12^0}{2}=-k\left(180^0\right)-6^0.$$

This is not valid because for any integer k, x is not acute.

If $n=2k+1$, then $x+10^0=(2k+1)\left(180^0\right)-22^0-3x$

$$\Rightarrow\ 4x=(2k+1)180^0-32^0\Rightarrow x=(2k+1)45^0-8^0.$$

Now, $k=0\ \Rightarrow\ x=37^0$, for other integral values of k, x is not acute.

$\therefore$ The only solution is $x=37^0$.

6. Problem: *Solve* $\cos 3\theta=\sin 2\theta$.

Solution: $\cos 3\theta=\sin 2\theta=\cos\left(\frac{\pi}{2}-2\theta\right)$

$$\Rightarrow\ 3\theta=2n\pi\pm\left(\frac{\pi}{2}-2\theta\right),\ n\in\mathbf{Z}$$

$$\Rightarrow\ 5\theta=(4n+1)\frac{\pi}{2}\ \text{ or }\ \theta=(4n-1)\frac{\pi}{2},\ n\in\mathbf{Z}$$

$$\Rightarrow\ \theta=(4n+1)\frac{\pi}{10},\ n\in\mathbf{Z}\ \text{ or }\ \theta=(4n-1)\frac{\pi}{2},\ n\in\mathbf{Z}.$$

7. Problem: *Solve* $7\sin^2\theta+3\cos^2\theta=4$.

Solution: Given that $7\sin^2\theta+3\cos^2\theta=4$

$$\Rightarrow\ 7\sin^2\theta+3\left(1-\sin^2\theta\right)=4$$

$$\Rightarrow\ 4\sin^2\theta=1\Rightarrow\sin\theta=\pm\frac{1}{2}.$$

$\therefore$ Principal solutions are $\theta=\pm\frac{\pi}{6}$,

and the general solution is given by $\theta=n\pi\pm\frac{\pi}{6},\ n\in\mathbf{Z}$.

8. Problem: *Solve* $2\cos^2\theta - \sqrt{3}\sin\theta + 1 = 0$.

Solution: $2\cos^2\theta - \sqrt{3}\sin\theta + 1 = 0$

$$\Rightarrow 2\sin^2\theta + \sqrt{3}\sin\theta - 3 = 0$$

$$\Rightarrow \left(2\sin\theta - \sqrt{3}\right)\left(\sin\theta + \sqrt{3}\right) = 0$$

$$\Rightarrow \sin\theta = \frac{\sqrt{3}}{2} \quad (\text{since } \sin\theta = -\sqrt{3} \text{ cannot happen})$$

$\therefore \theta = \frac{\pi}{3}$ is the principal solution and general solution is given by

$$\theta = n\pi + (-1)^n \frac{\pi}{3}, n \in \mathbf{Z}.$$

9. Problem: *Find all values of* $x \neq 0$ *in* $(-\pi, \pi)$ *satisfying the equation*

$$8^{1 + \cos x + \cos^2 x + \ldots} = 4^3.$$

Solution: For $x \neq 0$, we have $|\cos x| < 1$.

Then $1 + \cos x + \cos^2 x + \cdots = \frac{1}{1 - \cos x}$.

Now, $8^{1 + \cos x + \cos^2 x + \ldots} = 4^3 = 8^2 \Rightarrow 1 + \cos x + \cos^2 x + \cdots = 2$.

$$\Leftrightarrow \frac{1}{1 - \cos x} = 2 \Leftrightarrow \cos x = \frac{1}{2}$$

$$\Leftrightarrow x = \frac{\pi}{3} \text{ or } \frac{-\pi}{3} \quad (\text{since } x \in (-\pi, \pi)).$$

10. Problem: *Solve* $\tan\theta + 3\cot\theta = 5\sec\theta$.

Solution: First observe that the equation is valid only when $\cos\theta \neq 0$ and $\sin\theta \neq 0$.

Now $\tan\theta + 3\cot\theta = 5\sec\theta$

$$\Rightarrow \sin^2\theta + 3\cos^2\theta = 5\sin\theta$$

$$\Rightarrow \sin^2\theta + 3 - 3\sin^2\theta = 5\sin\theta$$

$$\Rightarrow 2\sin^2\theta + 5\sin\theta - 3 = 0$$

$$\Rightarrow (2\sin\theta - 1)(\sin\theta + 3) = 0$$

$$\Rightarrow \sin\theta = 1/2 \quad (\text{since } \sin\theta = -3 \text{ has no solution as } |\sin\theta| \leq 1 \text{ always})$$

$\therefore$ Principal solution is $\theta = \frac{\pi}{6}$ and general solution is $n\pi + (-1)^n \frac{\pi}{6}, n \in \mathbf{Z}$.

11. Problem: *Solve* $1 + \sin^2\theta = 3\sin\theta\cos\theta$.

Solution: Clearly $\cos\theta \neq 0$, so we divide both sides by $\cos^2\theta$, and we get

$$\sec^2\theta + \tan^2\theta = 3\tan\theta.$$

$$\Leftrightarrow\ 2\tan^2\theta - 3\tan\theta + 1 = 0$$

$$\Leftrightarrow\ (2\tan\theta - 1)(\tan\theta - 1) = 0$$

$$\Leftrightarrow\ \tan\theta = 1 \quad\text{or}\quad \tan\theta = \frac{1}{2}.$$

Now $\tan\theta = 1$ when $\theta = \frac{\pi}{4}$ and the general solution is $\theta = n\pi + \frac{\pi}{4},\ n\in \mathbf{Z}$.

Let α be the principal solution of $\tan\theta = \frac{1}{2}$.

Then the general solution is $\theta = n\pi + \alpha$.

12. Problem: *Solve* $\sqrt{2}\,(\sin x + \cos x) = \sqrt{3}$.

Solution: On dividing both sides by 2, we get

$$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{\sqrt{3}}{2}$$

$$\Rightarrow\ \sin\frac{\pi}{4}\sin x + \cos\frac{\pi}{4}\cos x = \frac{\sqrt{3}}{2}$$

$$\Rightarrow\ \cos\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2}$$

$\therefore$ The principal solution is $x - \frac{\pi}{4} = \frac{\pi}{6}$ i.e., $x = \frac{5\pi}{12}$.

The general solution is given by, $x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{6},\ n \in \mathbf{Z}$

or $x = 2n\pi + \frac{5\pi}{12}$ or $x = 2n\pi + \frac{\pi}{12},\ n \in \mathbf{Z}$.

13. Problem: *Find general solution of* θ *which satisfies both the equations*

$\sin\theta = -\frac{1}{2}$ *and* $\cos\theta = -\frac{\sqrt{3}}{2}$.

Solution: Given $\sin\theta = -\frac{1}{2} = -\sin\frac{\pi}{6}$

$$= \sin\left(\pi + \frac{\pi}{6}\right) = \sin\left(2\pi - \frac{\pi}{6}\right)$$

$$= \sin\frac{7\pi}{6} = \sin\frac{11\pi}{6}.$$

Therefore considering only angles in $(0, 2\pi)$, the only values of θ satisfying $\sin\theta = -\frac{1}{2}$ are $\frac{7\pi}{6}$ or $\frac{11\pi}{6}$

.

$$\cos\theta = -\frac{\sqrt{3}}{2} = -\cos\frac{\pi}{6}$$
$$= \cos\left(\pi - \frac{\pi}{6}\right) \text{ or } \cos\left(\pi + \frac{\pi}{6}\right)$$
$$= \cos\frac{5\pi}{6} \text{ or } \cos\frac{7\pi}{6}.$$

$\therefore$ Considering only angles in $(0, 2\pi)$, the only values of θ satisfying $\cos\theta = -\frac{\sqrt{3}}{2}$ are $\frac{5\pi}{6}$ or $\frac{7\pi}{6}$.

Thus $\frac{7\pi}{6}$ is the only angle which satisfies both the equations.

Hence general solution for θ is

$$\theta = 2n\pi + \frac{7\pi}{6}, \; n \in \mathbf{Z}.$$

14. Problem: *If* θ_1, θ_2 *are solutions of the equation* $a\cos 2\theta + b\sin 2\theta = c$, $\tan\theta_1 \neq \tan\theta_2$ *and* $a + c \neq 0$, *then find the values of*

(i) $\tan\theta_1 + \tan\theta_2$ (ii) $\tan\theta_1 . \tan\theta_2$

Solution: $a\cos 2\theta + b\sin 2\theta = c$

$$\Leftrightarrow \; a\left(\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right) + b\left(\frac{2\tan\theta}{1 + \tan^2\theta}\right) = c$$

$$\Leftrightarrow \; a - a\tan^2\theta + 2b\tan\theta = c + c\tan^2\theta$$

$$\Leftrightarrow \; (a + c)\tan^2\theta - 2b\tan\theta + (c - a) = 0 \qquad \text{... (1)}$$

This is a quadratic equation in $\tan\theta$. Since θ_1, θ_2 are the solutions of the given equation, we get that $\tan\theta_1, \tan\theta_2$ are the solutions of (1).

$$\therefore \; \tan\theta_1 + \tan\theta_2 = \frac{2b}{a + c}$$

$$\text{and } \tan\theta_1 . \tan\theta_2 = \frac{c - a}{c + a}.$$

15. Problem: *Solve* $4\sin x\ \sin 2x \sin 4x = \sin 3x$.

Solution: $\sin 3x = 2\sin x\,(2\sin 2x \sin 4x)$

$$= 2\sin x\,(\cos 2x - \cos 6x)$$

$$= 2\cos 2x \sin x - 2\cos 6x \sin x$$

$$= \sin 3x - \sin x - 2\cos 6x \sin x$$

$$\Rightarrow 2\cos 6x \sin x + \sin x = 0$$

$$\Rightarrow \sin x(2\cos 6x + 1) = 0$$

$$\Rightarrow \sin x = 0 \text{ or } \cos 6x = -\frac{1}{2}.$$

(i) $\sin x = 0 \Rightarrow x = 0$ is the principal solution and $x = n\pi,\ n \in \mathbf{Z}$ is the general solution.

(ii) $\cos 6x = -\frac{1}{2} \Rightarrow 6x = \frac{2\pi}{3}$ or $x = \frac{\pi}{9}$ is the principal solution.

The general solution is given by $6x = 2n\pi \pm \frac{2\pi}{3},\ n \in \mathbf{Z}$

$$x = \frac{n\pi}{3} \pm \frac{\pi}{9},\ n \in \mathbf{Z}.$$

16. Problem: *If* $0<\theta<\pi$, *solve* $\cos\theta . \cos 2\theta \cos 3\theta = \frac{1}{4}$.

Solution: $1 = 4\cos\theta \cos 2\theta \cos 3\theta$

$$= 2\cos 2\theta\,(2\cos 3\theta \cos\theta)$$

$$= 2\cos 2\theta\,(\cos 4\theta + \cos 2\theta)$$

$$= 2\cos 4\theta \cos 2\theta + 2\cos^2 2\theta$$

$$\Rightarrow 2\cos 4\theta \cos 2\theta + \cos 4\theta = 0$$

$$\Rightarrow \cos 4\theta\,(2\cos 2\theta + 1) = 0$$

$$\Rightarrow \cos 4\theta = 0 \text{ or } \cos 2\theta = -\frac{1}{2}.$$

(i) $\cos 4\theta = 0 \Rightarrow 4\theta = \frac{\pi}{2}$ is the principal solution and

$4\theta = 2n\pi \pm \frac{\pi}{2}$ is the general solution

so that $\theta = \frac{n\pi}{2} \pm \frac{\pi}{8},\ n \in \mathbf{Z}$ is the general solution

Put $n = 0, 1, 2$. We get $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$ are the solutions that lie in $(0, \pi)$.

(ii) $\cos 2\theta = -\frac{1}{2} \Rightarrow 2\theta = \frac{2\pi}{3}$ is the principal solution and

$$2\theta = 2n\pi \pm \frac{2\pi}{3},\ n\in \mathbf{Z} \text{ is the general solution}$$

$$\Rightarrow \theta = n\pi \pm \frac{\pi}{3},\ n\in \mathbf{Z} \text{ is the general solution}$$

Put $n = 0, 1$ we get $\frac{\pi}{3}, \frac{2\pi}{3}$ are the solutions that lie in the interval $(0, \pi)$. Hence the solutions of the given equation in $(0, \pi)$ are $\frac{\pi}{8}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{7\pi}{8}$.

17. Problem: *Given $p \neq \pm q$, show that the solutions of $\cos p\theta + \cos q\theta = 0$ form two series each of which is in A.P. Find also the common difference of each A.P.*

Solution: $\cos p\theta + \cos q\theta = 0$

$$\Rightarrow 2\cos\left[\left(\frac{p+q}{2}\right)\theta\right]\cos\left[\left(\frac{p-q}{2}\right)\theta\right] = 0$$

$$\Rightarrow \cos\left(\frac{p+q}{2}\right)\theta = 0 \text{ or } \cos\left(\frac{p-q}{2}\right)\theta = 0$$

$$\therefore \cos\left(\frac{p+q}{2}\right)\theta = 0 = \cos\frac{\pi}{2}$$

$$\Leftrightarrow \left(\frac{p+q}{2}\right)\theta = 2n\pi \pm \frac{\pi}{2} = (4n\pm 1)\frac{\pi}{2}$$

$$\Leftrightarrow \theta = \frac{(4n\pm 1)\pi}{(p+q)},\ n\in \mathbf{Z}$$

The solutions

$-\frac{\pi}{p+q}, \frac{\pi}{p+q}, \frac{3\pi}{p+q}, \frac{5\pi}{p+q}, \ldots$ form an A.P. with common difference $\frac{2\pi}{(p+q)}$.

Similarly the solutions of $\cos\left(\frac{p-q}{2}\right)\theta = 0$ are

$-\frac{\pi}{p-q}, \frac{\pi}{p-q}, \frac{3\pi}{p-q}, \frac{5\pi}{p-q}, \ldots$ which form another A.P. with common difference $\frac{2\pi}{(p-q)}$.

18. Problem: *Solve* $\sin 2x - \cos 2x = \sin x - \cos x$.

Solution: $(\sin 2x - \sin x) - (\cos 2x - \cos x) = 0$

$$\Rightarrow \quad 2\cos\frac{3x}{2}\sin\frac{x}{2} + 2\sin\frac{3x}{2}\sin\frac{x}{2} = 0$$

$$\Rightarrow \quad 2\sin\frac{x}{2}\left(\cos\frac{3x}{2} + \sin\frac{3x}{2}\right) = 0$$

$$\Rightarrow \quad \sin\frac{x}{2} = 0 \quad \text{or} \quad \cos\frac{3x}{2} + \sin\frac{3x}{2} = 0$$

$$\Rightarrow \quad \sin\frac{x}{2} = 0 \quad \text{or} \quad \tan\frac{3x}{2} = -1.$$

(i) $\sin\frac{x}{2} = 0 \;\Rightarrow\; \frac{x}{2} = n\pi,\ n \in \mathbf{Z} \;\Rightarrow\; x = 2n\pi,\ n \in \mathbf{Z}$.

(ii) $\tan\frac{3x}{2} = -1 \;\Rightarrow\; \frac{3x}{2} = -\frac{\pi}{4}$ is the principal solution and the general solution is

$$\frac{3x}{2} = n\pi - \frac{\pi}{4} \quad \text{or} \quad x = \frac{2n\pi}{3} - \frac{\pi}{6},\ n \in \mathbf{Z}.$$

$\therefore$ Solution set for the given equation is $\{2n\pi \mid n \in \mathbf{Z}\} \cup \left\{2n\frac{\pi}{3} - \frac{\pi}{6} \mid n \in \mathbf{Z}\right\}$.

Exercise 7(a)

I. 1. Find the principal solutions of the angles in the equations

(i) $2\cos^2\theta = 1$ (ii) $\sqrt{3}\sec\theta + 2 = 0$

(iii) $3\tan^2\theta = 1$

2. Solve the following equations

(i) $\cos 2\theta = \frac{\sqrt{5}+1}{4},\ \theta \in [0, 2\pi]$ (ii) $\tan^2\theta = 1,\ \theta \in [-\pi, \pi]$

(iii) $\sin 3\theta = \frac{\sqrt{3}}{2},\ \theta \in [-\pi, \pi]$ (iv) $\cos^2\theta = \frac{3}{4},\ \theta \in [0, \pi]$

(v) $2\sin^2\theta = \sin\theta,\ \theta \in (0, \pi)$

3. Find general solutions of the following equations.

(i) $\sin\theta = \frac{\sqrt{3}}{2},\ \cos\theta = -\frac{1}{2}$ (ii) $\tan x = -\frac{1}{\sqrt{3}},\ \sec x = \frac{2}{\sqrt{3}}$

(iii) $\operatorname{cosec}\theta = -2,\ \cot\theta = -\sqrt{3}$

4. (i) If $\sin(270^0 - x) = \cos 292^0$, then find x in $(0, 360^0)$.

(ii) If $x < 90^0$ and $\sin(x + 28^0) = \cos(3x - 78^0)$, then find x.

5. Find general solutions of the following equations.

(i) $2\sin^2\theta = 3\cos\theta$. (ii) $\sin^2\theta - \cos\theta = \frac{1}{4}$

(iii) $5\cos^2\theta + 7\sin^2\theta = 6$ (iv) $3\sin^4 x + \cos^4 x = 1$

II. 1. Solve the following equations and write general solutions.

(i) $2\sin^2\theta - 4 = 5\cos\theta$ (ii) $2 + \sqrt{3}\sec x - 4\cos x = 2\sqrt{3}$

(iii) $2\cos^2\theta + 11\sin\theta = 7$ (iv) $6\tan^2 x - 2\cos^2 x = \cos 2x$

(v) $4\cos^2\theta + \sqrt{3} = 2(\sqrt{3}+1)\cos\theta$ (vi) $1 + \sin 2x = (\sin 3x - \cos 3x)^2$

(vii) $2\sin^2 x + \sin^2 2x = 2$

2. Solve the following equations

(i) $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$ (ii) $\cot x + \operatorname{cosec} x = \sqrt{3}$

(iii) $\sin x + \sqrt{3}\cos x = \sqrt{2}$

3. Solve the following equations

(i) $\tan\theta + \sec\theta = \sqrt{3},\ 0 \le \theta \le 2\pi$

(ii) $\cos 3x + \cos 2x = \sin\frac{3x}{2} + \sin\frac{x}{2};\ 0 \le x \le 2\pi$

(iii) $\cot^2 x - (\sqrt{3}+1)\cot x + \sqrt{3} = 0;\ 0 < x < \frac{\pi}{2}$

(iv) $\sec x \cdot \cos 5x + 1 = 0;\ 0 < x < 2\pi$

III. 1. (i) Solve $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$.

(ii) If $x + y = \frac{2\pi}{3}$ and $\sin x + \sin y = \frac{3}{2}$ find x and y.

(iii) If $\sin 3x + \sin x + 2\cos x = \sin 2x + 2\cos^2 x$, find the general solution.

(iv) Solve $\cos 3x - \cos 4x = \cos 5x - \cos 6x$.

2. Solve the following equations.

(i) $\cos 2\theta + \cos 8\theta = \cos 5\theta$ (ii) $\cos\theta - \cos 7\theta = \sin 4\theta$

(iii) $\sin\theta + \sin 5\theta = \sin 3\theta,\ 0 < \theta < \pi$.

3. (i) If $\tan p\theta = \cot q\theta$, and $p \ne -q$ show that the solutions are in A.P. with common difference $\frac{\pi}{p+q}$.

(ii) Show that the solutions of $\cos p\theta = \sin q\theta$ form two series each of which is an A.P. Find also the common difference of each A.P. ($p \neq \pm q$)

(iii) Find the number of solutions of the equation $\tan x + \sec x = 2\cos x$; $\cos x \neq 0$, lying in the interval $(0, \pi)$.

(iv) Solve $\sin 3\alpha = 4 \sin\alpha \sin(x+\alpha)\sin(x-\alpha)$ where $\alpha \neq n\pi, n \in \mathbf{Z}$.

4. (i) If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$, then prove that $\cos\left(\theta - \frac{\pi}{4}\right) = \pm\frac{1}{2\sqrt{2}}$.

(ii) Find the range of θ if $\cos\theta + \sin\theta$ is positive.

5. If α, β are the solutions of the equation $a\cos\theta + b\sin\theta = c$, where $a, b, c \in \mathbf{R}$ and if $a^2 + b^2 > 0$, $\cos\alpha \neq \cos\beta$ and $\sin\alpha \neq \sin\beta$ then show that

(i) $\sin\alpha + \sin\beta = \dfrac{2bc}{a^2+b^2}$ (ii) $\cos\alpha + \cos\beta = \dfrac{2ac}{a^2+b^2}$

(iii) $\cos\alpha.\cos\beta = \dfrac{c^2-b^2}{a^2+b^2}$ (iv) $\sin\alpha.\sin\beta = \dfrac{c^2-a^2}{a^2+b^2}$

6. (i) Find the common roots of the equations $\cos 2x + \sin 2x = \cot x$ and $2\cos^2 x + \cos^2 2x = 1$.

(ii) Solve the equation $\sqrt{6 - \cos x + 7\sin^2 x} + \cos x = 0$.

(iii) If $|\tan x| = \tan x + \dfrac{1}{\cos x}$ and $x \in [0, 2\pi]$, find the value of x.

Key Concepts

- If $k \in [-1, 1]$, then the principal solution θ of $\sin x = k$ lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the general solution is given by $n\pi + (-1)^n\theta$, $n \in \mathbf{Z}$.
- If $k \in [-1, 1]$, then the principal solution θ of $\sin x = k$ lies in $[0, \pi]$ and the general solution is given by $2n\pi \pm \theta$, $n \in \mathbf{Z}$.
- If $k \in \mathbf{R}$, then the principal solution θ of $\tan x = k$ lies in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and the general solution is given by $n\pi \pm \theta$, $n \in \mathbf{Z}$.
- If $k \in (-\infty, -1] \cup [1, \infty)$, then the principal solution θ of $\operatorname{cosec} x = k$ lies in $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$ and the general solution is given by $n\pi + (-1)^n\theta$, $n \in \mathbf{Z}$.

- If $k \in (-\infty, -1] \cup [1, \infty)$, then the principal solution θ of $\sec x = k$ lies in $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ and the general solution is given by $2n\pi \pm \theta$, $n \in \mathbf{Z}$.
- If $k \in \mathbf{R}$, then the principal solution θ of $\cot x = k$ lies in $(0, \pi)$ and the general solution is given by $n\pi + \theta$, $n \in \mathbf{Z}$.
- If $k \in [0, 1]$, then the general solution of the equation $\sin^2 x = k$ is $n\pi \pm \alpha$, $n \in \mathbf{Z}$ (where $\sin^2 \alpha = k$).
- If $k \in [0, 1]$, then the general solution of the equation $\cos^2 x = k$ is $n\pi \pm \alpha$, $n \in \mathbf{Z}$ (where $\cos^2 \alpha = k$).

  Similarly, we get that the general solutions of the equations $\tan^2 x = k$, $\cot^2 x = k$, $\sec^2 x = k$, $\text{cosec}^2 x = k$ are of the form $n\pi \pm \alpha$ whenever a solution exists.
- The equation $a \sin x + b \cos x = c$ $(a, b, c \in \mathbf{R}$ and $a^2 + b^2 \neq 0)$ has a solution if and only if $c^2 \leq a^2 + b^2$.

Historical Note

Brahmagupta (7th century A.D.) is one of the most celebrated mathematicians of ancient India. He wrote a standard treatise "Brahmasphuta siddhanth" on ancient Indian Astronomy. *Brahmagupta* is famous for many contributions to astronomy, trigonometry, algebra and geometry. The simple rule to help the memory for the sine function, $\sqrt{(0, 1, 2, 3, 4)/4} = \sin(0, 30, 45, 60, 90)^0$ is found in the works of *Brahmagupta*. This shows the level of advancement of trigonometry in those days.

The historian *al-Biruni* (ca - 1050) in his book Tariq al-Hind states that the *Abbasid Caliph al-Ma'mun* had an embassy in India and from India a book was brought to Baghdad which was translated into Arabic as Sindhind. It is generally presumed that Sindhind is none other than *Brahmagupta's* Brahmasphuta-Siddhanta.

Bhaskaracharya hailed *Brahmagupta* as "Ganakachakra Chudamani", precious jewel amidst the circle of Mathematicians.

Answers

Exercise 7(a)

I. 1. (i) $45^0, 135^0$ (ii) 150^0 (iii) $\pm\pi/6$

2. (i) $\frac{\pi}{10}, \frac{9\pi}{10}, \frac{11\pi}{10}, \frac{19\pi}{10}$ (ii) $\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$

(iii) $-\frac{5\pi}{9}, -\frac{4\pi}{9}, \frac{\pi}{9}, \frac{2\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}$ (iv) $\frac{\pi}{6}, \frac{5\pi}{6}$ (v) $\frac{\pi}{6}, \frac{5\pi}{6}$

3. (i) $2n\pi+\frac{2\pi}{3}, n\in \mathbf{Z}$ (ii) $2n\pi-\frac{\pi}{6}, n\in \mathbf{Z}$

(iii) $2n\pi-\frac{\pi}{6}, n\in \mathbf{Z}$

4. (i) $112^0, 248^0$ (ii) $8^0, 35^0$

5. (i) $2n\pi\pm\frac{\pi}{3}, n\in \mathbf{Z}$ (ii) $2n\pi\pm\frac{\pi}{3}, n\in \mathbf{Z}$

(iii) $n\pi\pm\frac{\pi}{4}, n\in \mathbf{Z}$ (iv) $n\pi$ or $n\pi\pm\frac{\pi}{4}, n\in \mathbf{Z}$

II. 1. (i) $2n\pi\pm\frac{2\pi}{3}, n\in \mathbf{Z}$ (ii) $2n\pi\pm\frac{\pi}{3}$ or $2n\pi\pm\frac{5\pi}{6}, n\in \mathbf{Z}$

(iii) $n\pi+(-1)^n\frac{\pi}{6}, n\in \mathbf{Z}$ (iv) $n\pi\pm\frac{\pi}{6}, n\in \mathbf{Z}$

(v) $2n\pi\pm\frac{\pi}{3}$ or $2n\pi\pm\frac{\pi}{6}, n\in \mathbf{Z}$ (vi) $\frac{n\pi}{4}$ or $(2n+1)\frac{\pi}{4}, n\in \mathbf{Z}$

(vii) $(2n+1)\frac{\pi}{2}, n\pi\pm\frac{\pi}{4}, n\in \mathbf{Z}$

2. (i) $\theta = n\pi+(-1)^n\frac{\pi}{4}+\frac{\pi}{6}, n\in \mathbf{Z}$ (ii) $2n\pi+\frac{\pi}{3}, n\in \mathbf{Z}$

(iii) $2n\pi+\frac{5\pi}{12}, 2n\pi-\frac{\pi}{12}, n\in \mathbf{Z}$

3. (i) $\frac{\pi}{6}$ (ii) $\frac{\pi}{7}, \frac{5\pi}{7}, \pi, \frac{9\pi}{7}, \frac{13\pi}{7}$ (iii) $\frac{\pi}{6}, \frac{\pi}{4}$

(iv) $\frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{5\pi}{4}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{7\pi}{4}$

III. 1. (i) $2n\pi\pm\frac{2\pi}{3}, \frac{n\pi}{2}+\frac{\pi}{8}, n\in \mathbf{Z}$

(ii) $x = 2n\pi+\frac{\pi}{2}, y = \frac{\pi}{6}-2n\pi$ or $x = 2n\pi+\frac{\pi}{6}, y = \frac{\pi}{2}-2n\pi, n\in \mathbf{Z}$

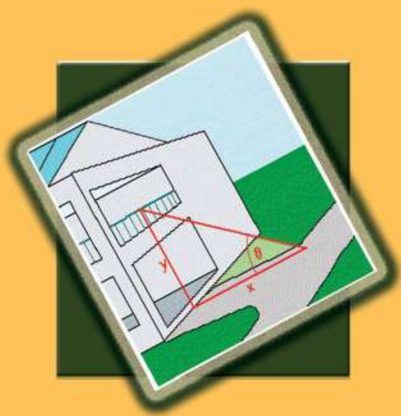
(iii) $(2n+1)\frac{\pi}{2}, n\pi-\frac{\pi}{4}, 2n\pi+\frac{\pi}{2}, 2n\pi, n\in \mathbf{Z}$ (iv) $(2n+1)\frac{\pi}{9}, n\pi, n\in \mathbf{Z}$

2. (i) $(2n+1)\frac{\pi}{10}, \frac{2n\pi}{3}\pm\frac{\pi}{9}, n\in \mathbf{Z}$ (ii) $\frac{n\pi}{4}$ or $\frac{n\pi}{3}+(-1)^n\frac{\pi}{18}$ (iii) $\frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}$

3. (ii) $\frac{2\pi}{p\pm q}$ (iii) 2 (iv) $n\pi\pm\frac{\pi}{3}$

4. (ii) $\bigcup_{n\in \mathbf{Z}}\left(2n\pi-\frac{\pi}{4}, 2n\pi+\frac{3\pi}{4}\right)$

6. (i) $(2n+1)\frac{\pi}{4}, n\in \mathbf{Z}$ (ii) No solution (iii) $\frac{11\pi}{6}$



Chapter 8

Inverse Trigonometric Functions

"If there be light, then there is darkness; if cold, heat; if height, depth, if solid, fluid; if hard, soft; if rough, smooth; if calm, tempest; if prosperity, adversity; if life, death"

– Pythagoras

Introduction

Let us recall that if A, B are sets and $f : A \to B$ is a bijection, then $g : B \to A$ is the inverse of the function f if $g \circ f = I_A$ (Indentity on A) and $f \circ g = I_B$.

This function g is unique and it is denoted by f^{-1}.

Equivalently, a function $f : A \to B$ has inverse if and only if f is a bijection. The inverse $f^{-1} : B \to A$ of f is defined by $f^{-1}(x) = y$, where $f(y) = x$.

We have come across many functions which possess inverse and functions which do not possess inverse. All trigonometric functions possess inverses if we take the domain suitably. In this chapter, we learn about inverse functions of all trigonometric functions.

James Gregory

(1638 - 1675)

Gregory *was professor of mathematics at St. Andrews and at Edinburgh. He was equally interested in physics and published a work on optics in which he described the reflecting telescope, now known by his name. In mathematics he expanded functions in infinite series and was one of the first who distinguished between convergent and divergent series.*

8.1 To reduce a trigonometric function into a bijective function

Let us consider the function $f : \mathbf{R} \to [-1, 1]$ defined by $f(x) = \sin x$, for all $x \in \mathbf{R}$.

This function f is a surjection, but not an injection on **R** since $f(2n\pi + x) = f(x)$ for all $n \in \mathbf{Z}$ and $x \in \mathbf{R}$ which means that, for any $t \in [-1, 1]$, there are infinitely many $x \in \mathbf{R}$ such that $f(x) = t$. But, for any $t \in [-1, 1]$, there exists **unique** $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ such that $f(x) = t$.

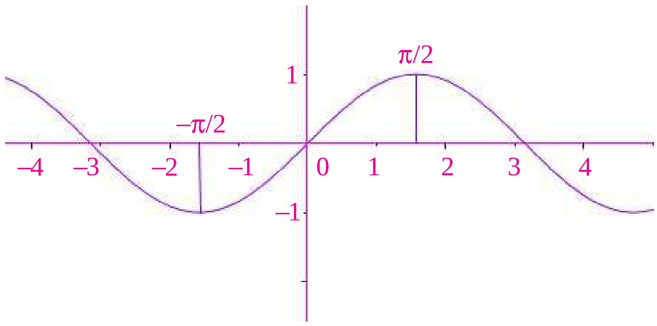


Fig. 8.1

Also $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is a subinterval of largest length on which sine function is a bijection (see Fig. 8.1).

In other words, the function $g : \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ defined by $g(x) = \sin x$ for all $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is a bijection and hence it has inverse. The inverse g^{-1} of g is a function from $[-1, 1]$ onto $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

We denote this function g by sin and its inverse g^{-1} by Sin^{-1} or arc sin.

8.1.1 Definition: *The function* $\text{Sin}^{-1} : [-1, 1] \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ *defined by*

$$\text{Sin}^{-1} x = \theta \Leftrightarrow \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \text{ and } \sin\theta = x,$$

for all $x \in [-1, 1]$, *is called the* ***'inverse sine function'***. *This function is also denoted by 'arc sin'.*

8.1.2 Note: If $x \in [-1, 0)$, then $\text{Sin}^{-1} x \in \left[\frac{-\pi}{2}, 0\right)$ and if $x \in (0, 1]$, then $\text{Sin}^{-1} x \in \left(0, \frac{\pi}{2}\right]$ and $\text{Sin}^{-1} 0 = 0$. We can define the inverse trigonometric functions of cosine, tangent, cotangent, secant, cosecant similarly by taking the domain suitably as given below.

8.1.3 Definitions

(i) *The function* $\text{Cos}^{-1} : [-1, 1] \to [0, \pi]$ *is defined for all* $x \in [-1, 1]$ *by* $\text{Cos}^{-1} x = \theta$ *if and only if* $\theta \in [0, \pi]$ *and* $\cos\theta = x$.

(ii) *The function* $\text{Tan}^{-1} : \mathbf{R} \to \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ *is defined for all* $x \in \mathbf{R}$ *by* $\text{Tan}^{-1} x = \theta$ *if and only if* $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ *and* $\tan\theta = x$.

(iii) *The function* $\text{Sec}^{-1} : (-\infty, -1] \cup [1, \infty) \to \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ *is defined for all* $x \in (-\infty, -1] \cup [1, \infty)$ *by* $\text{Sec}^{-1} x = \theta$ *if and only if* $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ *and* $\sec\theta = x$.

(iv) *The function* $\text{Cosec}^{-1} : (-\infty, -1] \cup [1, \infty) \to \left[\frac{-\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ *is defined for all* $x \in (-\infty, -1] \cup [1, \infty)$ *by* $\text{Cosec}^{-1} x = \theta$ *if and only if* $\theta \in \left[\frac{-\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ *and* $\text{cosec}\,\theta = x$.

(v) *The function* $\text{Cot}^{-1} : \mathbf{R} \to (0, \pi)$ *is defined for all* $x \in \mathbf{R}$ *by* $\text{Cot}^{-1}(x) = \theta$ *if and only if* $\theta \in (0, \pi)$ *and* $\cot\theta = x$.

All the above facts can be understood easily from the following table.

8.1.4 Domains and ranges of the inverse trigonometric functions

Table 8.1

Inverse trigonometric function $y = f(x)$	Domain (x)	Range (y)
$y = \text{Sin}^{-1} x$	[-1, 1]	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$y = \text{Cos}^{-1} x$	[-1, 1]	$[0, \pi]$
$y = \text{Tan}^{-1} x$	**R**	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$y = \text{Cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[\frac{-\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
$y = \text{Sec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$y = \text{Cot}^{-1} x$	**R**	$(0, \pi)$

8.2 Graphs of Inverse trigonometric functions

In Chapter 6, we have given the graphs of all the six trigonometric functions.

Now we draw the graphs of the six inverse trigonometric functions by taking the domain on X-axis and range on Y-axis.

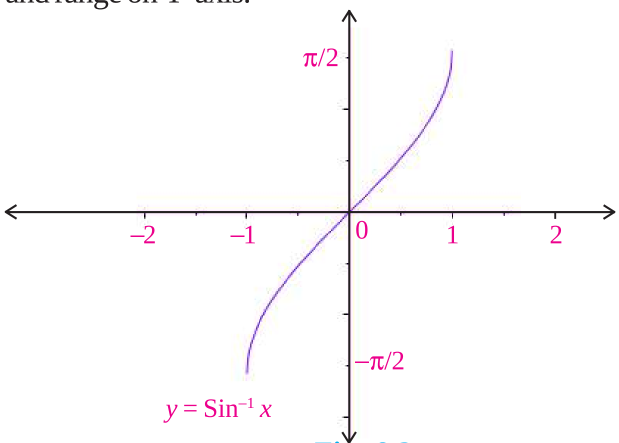


Fig. 8.2

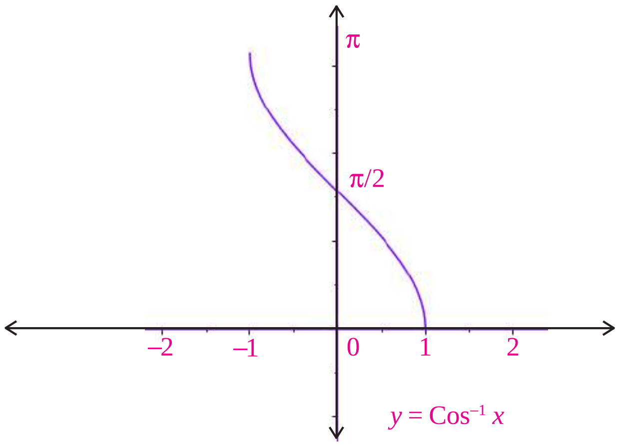


Fig. 8.3

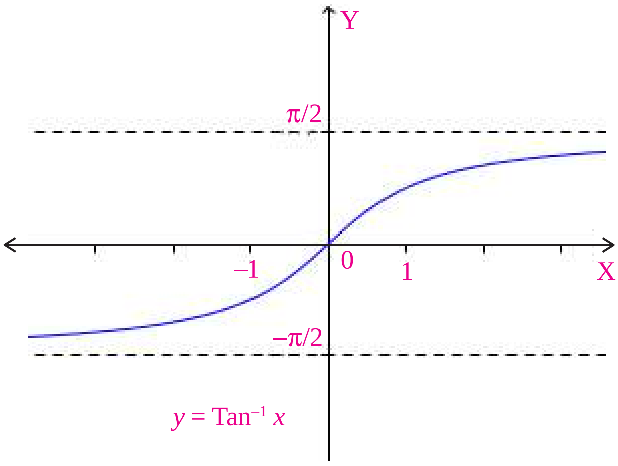


Fig. 8.4

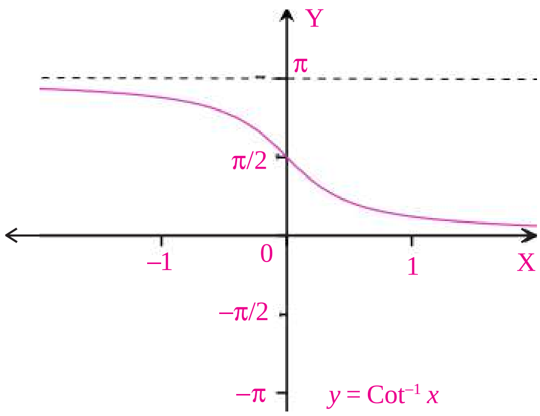


Fig. 8.5

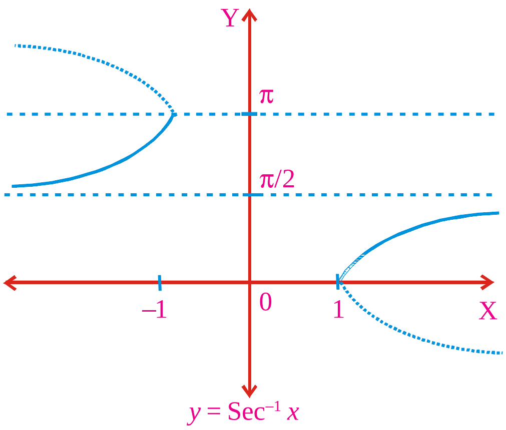


Fig. 8.6

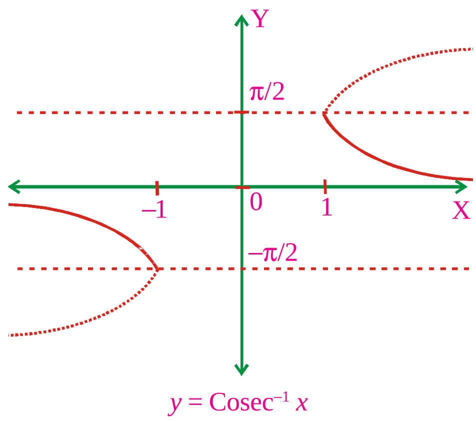


Fig. 8.7

8.3 Properties of inverse trigonometric functions

In this section we learn some elementary properties of inverse trigonometric functions defined in 8.1. These properties will be useful to solve easily many problems on inverse trigonometric functions.

8.3.1 Theorem

(i) *For* $x \in [-1, 0) \cup (0, 1]$, $\text{Sin}^{-1} x = \text{Cosec}^{-1}\frac{1}{x}$.

(ii) *For* $x \in [-1, 0) \cup (0, 1]$, $\text{Cos}^{-1} x = \text{Sec}^{-1}\frac{1}{x}$.

(iii) *For* $x > 0$, $\text{Tan}^{-1} x = \text{Cot}^{-1}\frac{1}{x}$.

(iv) *For* $x < 0$, $\text{Tan}^{-1} x = \text{Cot}^{-1}\frac{1}{x} - \pi$.

Proof

(i) Let $x \in [-1, 0) \cup (0, 1]$ and suppose $\text{Sin}^{-1}x = \theta$. Then $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$,

$\sin\theta = x$ and $\theta \neq 0$. Hence $\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{x}$.

Therefore, $\theta = \text{Cosec}^{-1}\frac{1}{x}$ or $\text{Sin}^{-1}x = \text{Cosec}^{-1}\frac{1}{x}$.

(ii) We can prove this as above.

(iii) Let $x \in (0, \infty)$ and suppose $\text{Tan}^{-1}x = \theta$.

Then $\theta \in \left(0, \frac{\pi}{2}\right)$ and $\tan\theta = x$. Then $\cot\theta = \frac{1}{x}$ and $\theta \in \left(0, \frac{\pi}{2}\right)$.

$$\therefore \text{Cot}^{-1}\frac{1}{x} = \theta = \text{Tan}^{-1}x.$$

(iv) Now, let $x \in (-\infty, 0)$ and $\text{Tan}^{-1}x = \theta$. Then $\theta \in \left(-\frac{\pi}{2}, 0\right)$ and $\tan\theta = x$.

That is,

$$\theta + \pi \in \left(\frac{\pi}{2}, \pi\right) \text{ and } \tan(\pi + \theta) = x.$$

Therefore, $\theta + \pi \in \left(\frac{\pi}{2}, \pi\right)$ and $\cot\theta = \frac{1}{x}$.

$$\therefore \text{Cot}^{-1}\frac{1}{x} = \theta + \pi = \pi + \text{Tan}^{-1}\frac{1}{x} \quad \text{or} \quad \text{Tan}^{-1}\frac{1}{x} = \text{Cot}^{-1}\frac{1}{x} - \pi.$$

8.3.2 Theorem

(i) *For* $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, $\text{Sin}^{-1}(\sin\theta) = \theta$,

(ii) *For* $x \in [-1, 1]$, $\sin\left(\text{Sin}^{-1}x\right) = x$.

Proof

(i) Let $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and write $x = \sin\theta$. Then $x \in [-1, 1]$.

Hence $\text{Sin}^{-1}x = \theta$. That is $\text{Sin}^{-1}(\sin\theta) = \theta$

(ii) Let $x \in [-1, 1]$ and suppose $\text{Sin}^{-1}x = \theta$. Then

$$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ and } \sin\theta = x. \text{ That is, } \sin\left(\text{Sin}^{-1}x\right) = x.$$

We can prove similar results for other inverse trigonometric functions also. We state them in the following theorem without proof.

8.3.3 Theorem

1. *If* $\theta \in [0, \pi]$, *then* $\text{Cos}^{-1}(\cos\theta) = \theta$ *and if* $x \in [-1, 1]$ *then*

   $\cos\left(\text{Cos}^{-1}x\right) = x$.

2. *If* $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, *then* $\text{Tan}^{-1}(\tan\theta) = \theta$ *and if* $x \in \mathbf{R}$, *then*

   $\tan\left(\text{Tan}^{-1}x\right) = x$.

3. *If* $\theta \in (0, \pi)$, *then* $\text{Cot}^{-1}(\cot\theta) = \theta$ *and if* $x \in \mathbf{R}$, *then* $\cot\left(\text{Cot}^{-1}x\right) = x$.

4. *If* $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, *then* $\text{Sec}^{-1}(\sec\theta) = \theta$ and

   If $x \in (-\infty, -1] \cup [1, \infty)$, *then* $\sec\left(\text{Sec}^{-1}x\right) = x$.

5. *If* $\theta \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, *then* $\text{Cosec}^{-1}(\text{cosec}\,\theta) = \theta$ and

   if $x \in (-\infty, -1] \cup [1, \infty)$, *then* $\text{cosec}\left(\text{Cosec}^{-1}x\right) = x$.

In the following we check which of the trigonometric functions are odd or even or neither. In otherwords we find the formulae for $f(-x)$ where f is an inverse trigonometric function.

8.3.4 Theorem

1. *If* $x \in [-1, 1]$, *then* $\text{Sin}^{-1}(-x) = -\text{Sin}^{-1}x$.
2. *If* $x \in [-1, 1]$, *then* $\text{Cos}^{-1}(-x) = \pi - \text{Cos}^{-1}x$.
3. *If* $x \in \mathbf{R}$, *then* $\text{Tan}^{-1}(-x) = -\text{Tan}^{-1}x$.
4. *If* $x \in \mathbf{R}$, *then* $\text{Cot}^{-1}(-x) = \pi - \text{Cot}^{-1}x$.
5. *For* $x \in (-\infty, -1] \cup [1, \infty)$, $\text{Sec}^{-1}(-x) = \pi - \text{Sec}^{-1}x$.
6. *For* $x \in (-\infty, -1] \cup [1, \infty)$, $\text{Cosec}^{-1}(-x) = -\text{Cosec}^{-1}x$.

Proof

1. Let $x \in [-1, 1]$, then $-x \in [-1, 1]$. If $\text{Sin}^{-1}(-x) = \theta$, then $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

and $-x = \sin\theta$. So that $x = -\sin\theta = \sin(-\theta)$ and $-\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Hence $\text{Sin}^{-1}x = -\theta$.

Thus $\theta = -\text{Sin}^{-1}x$. Therefore $\text{Sin}^{-1}(-x) = -\text{Sin}^{-1}x$.

2. If $x \in [-1, 1]$, then $-x \in [-1, 1]$. Let $\text{Cos}^{-1}(-x) = \theta$. Then $0 \le \theta \le \pi$ and

$-x = \cos\theta$. So that $x = -\cos\theta = \cos(\pi - \theta)$ and $0 \le \pi - \theta \le \pi$.

Thus $\text{Cos}^{-1}x = \pi - \theta$. That is, $\theta = \pi - \text{Cos}^{-1}x$. Hence

$$\text{Cos}^{-1}(-x) = \pi - \text{Cos}^{-1}x.$$

Similarly, we can prove (3) to (6).

8.3.5 Theorem

1. *If* $\theta \in [0, \pi]$, *then* $\text{Sin}^{-1}(\cos\theta) = \frac{\pi}{2} - \theta$.
2. *If* $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, *then* $\text{Cos}^{-1}(\sin\theta) = \frac{\pi}{2} - \theta$.
3. *If* $\theta \in (0, \pi)$, *then* $\text{Tan}^{-1}(\cot\theta) = \frac{\pi}{2} - \theta$.
4. *If* $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, *then* $\text{Cot}^{-1}(\tan\theta) = \frac{\pi}{2} - \theta$.
5. *If* $\theta \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, *then* $\text{Sec}^{-1}(\text{cosec}\,\theta) = \frac{\pi}{2} - \theta$.
6. *If* $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, *then* $\text{Cosec}^{-1}(\sec\theta) = \frac{\pi}{2} - \theta$.

Proof

1. Let $\theta \in [0, \pi]$. Then $-1 \le \cos\theta \le 1$. Now

$$\text{Sin}^{-1}(\cos\theta) = \text{Sin}^{-1}\left(\sin\left(\frac{\pi}{2} - \theta\right)\right) \text{ and } \frac{\pi}{2} - \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right].$$

Hence $\text{Sin}^{-1}(\cos\theta) = \frac{\pi}{2} - \theta$, by Theorem 8.3.3(1).

2. Let $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then $-1 \le \sin\theta \le 1$. So that

$$\text{Cos}^{-1}(\sin\theta) = \text{Cos}^{-1}\left(\cos\left(\frac{\pi}{2} - \theta\right)\right) \text{ and } \frac{\pi}{2} - \theta \in [0, \pi].$$

Hence $\text{Cos}^{-1}(\sin\theta) = \frac{\pi}{2} - \theta$, by Theorem 8.3.3 (1).

Similarly, we can prove the remaining results.

8.3.6 Theorem

1. $\text{Sin}^{-1}x = \text{Cos}^{-1}\sqrt{1-x^2}$ *if* $0 \le x \le 1$.
2. $\text{Sin}^{-1}x = -\text{Cos}^{-1}\sqrt{1-x^2}$ *if* $-1 \le x < 0$.

Proof

1. Let $0 \le x \le 1$ and $\text{Sin}^{-1}x = \theta$. Then $0 \le \theta \le \frac{\pi}{2}$.

Now $\sin\theta = x$ and hence $\cos\theta = \sqrt{1-x^2}$ and $0 \le \sqrt{1-x^2} \le 1$.

Therefore $\text{Cos}^{-1}\sqrt{1-x^2} = \theta = \text{Sin}^{-1}x$.

2. Suppose $-1 \le x < 0$ and $\text{Sin}^{-1}x = \theta$. Then $-\frac{\pi}{2} \le \theta < 0$.

So that $\sin\theta = x$ and $\cos\theta = \sqrt{1-x^2}$ (since $\cos\theta > 0$). Now

$$\cos(-\theta) = \sqrt{1-x^2} \text{ and } 0 < -\theta \le \frac{\pi}{2}.$$

Hence $\text{Cos}^{-1}\left(\sqrt{1-x^2}\right) = -\theta = -\text{Sin}^{-1}x$. Therefore, $\text{Sin}^{-1}x = -\text{Cos}^{-1}\sqrt{1-x^2}$.

We can prove the following theorem as above.

8.3.7 Theorem

1. $\text{Sin}^{-1}x = \text{Tan}^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ *if* $x \in (-1, 1)$.
2. $\text{Cos}^{-1}x = \text{Sin}^{-1}\sqrt{1-x^2}$ *if* $x \in [0, 1]$.
3. $\text{Cos}^{-1}x = \pi - \text{Sin}^{-1}\sqrt{1-x^2}$ *if* $x \in [-1, 0)$.

4. $\text{Tan}^{-1} x = \text{Sin}^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \text{Cos}^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$ *for* $x > 0$.

Now we prove an important theorem.

8.3.8 Theorem

1. $\text{Cos}^{-1}x + \text{Sin}^{-1}x = \frac{\pi}{2}$ *for all* $x \in [-1, 1]$.

2. $\text{Tan}^{-1}x + \text{Cot}^{-1}x = \frac{\pi}{2}$ *for all* $x \in \mathbf{R}$.

3. $\text{Sec}^{-1}x + \text{Cosec}^{-1}x = \frac{\pi}{2}$ *for all* $x \in (-\infty, -1] \cup [1, \infty)$.

Proof

1. Let $x \in [-1, 1]$ and $\text{Sin}^{-1}x = \theta$. Then $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin\theta = x$. Now

$x = \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ and $\frac{\pi}{2} - \theta \in [0, \pi]$. So that

$\text{Cos}^{-1}x = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \text{Sin}^{-1}x$.

Therefore, $\text{Cos}^{-1}x + \text{Sin}^{-1}x = \frac{\pi}{2}$ for all $x \in [-1, 1]$.

2. Let $x \in \mathbf{R}$ and $\text{Tan}^{-1}x = \theta$. Then $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\tan\theta = x$.

Now, $x = \tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$ and $0 < \frac{\pi}{2} - \theta < \pi$.

Thus $\text{Cot}^{-1}x = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \text{Tan}^{-1}x$.

Hence $\text{Tan}^{-1}x + \text{Cot}^{-1}x = \frac{\pi}{2}$ for all $x \in \mathbf{R}$.

3. Let $x \in (-\infty, -1] \cup [1, \infty)$ and $\text{Sec}^{-1}x = \theta$. Then $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and

$\sec\theta = x$. Now

$$x = \sec\theta = \text{cosec}\left(\frac{\pi}{2} - \theta\right) \text{ and } \frac{\pi}{2} - \theta \in \left[\frac{-\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right].$$

Thus $\text{Cosec}^{-1}x = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \text{Sec}^{-1}x$. Therefore,

$$\text{Sec}^{-1}x + \text{Cosec}^{-1}x = \frac{\pi}{2} \text{ for all } x \in (-\infty, -1] \cup [1, \infty).$$

8.3.9 Theorem

If $x \geq 0,\ y \geq 0$ *and* $x^2 + y^2 \leq 1$, *then*

$$\mathrm{Sin}^{-1} x + \mathrm{Sin}^{-1} y = \mathrm{Sin}^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}.$$

Proof

Suppose $x \geq 0,\ y \geq 0$ and $x^2 + y^2 \leq 1$. Let $\mathrm{Sin}^{-1} x = \alpha$ and $\mathrm{Sin}^{-1} y = \beta$.

Then $\alpha, \beta \in \left[0, \frac{\pi}{2}\right]$ and $\sin\alpha = x, \sin\beta = y$. So $\cos\alpha = \sqrt{1-x^2}$ and

$\cos\beta = \sqrt{1-y^2}$ (since $\cos\alpha, \cos\beta$ are non negative as $\alpha, \beta \in \left[0, \frac{\pi}{2}\right]$).

Now $0 \leq \alpha + \beta \leq \pi$ ... (1)

Also $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \sqrt{1-x^2}\sqrt{1-y^2} - xy \qquad \text{... (2)}$$

Now $x^2 + y^2 \leq 1 \Rightarrow 1 - x^2 - y^2 \geq 0 \Rightarrow 1 - x^2 - y^2 + x^2y^2 \geq x^2y^2$

$$\Rightarrow (1-x^2)(1-y^2) \geq x^2y^2$$

$$\Rightarrow \sqrt{(1-x^2)}\sqrt{1-y^2} \geq xy \qquad (\text{since } xy \geq 0).$$

Hence from (2), $\cos(\alpha+\beta) \geq 0$. So $0 \leq \alpha + \beta \leq \frac{\pi}{2}$ from (1)

Now $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= x\sqrt{1-y^2} + \sqrt{1-x^2}\,y.$$

Hence $\alpha + \beta = \mathrm{Sin}^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

or $\mathrm{Sin}^{-1} x + \mathrm{Sin}^{-1} y = \mathrm{Sin}^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$.

We can prove the following formulae also as above. We state them without proof.

8.3.10 Theorem

1. *If* $-1 \leq x, y \leq 1,\ xy < 0$ *and* $x^2 + y^2 > 1$, *then*

$$\mathrm{Sin}^{-1} x + \mathrm{Sin}^{-1} y = \mathrm{Sin}^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right).$$

2. *If* $0 < x, y \leq 1$, *and* $x^2 + y^2 > 1$, *then*

$$\mathrm{Sin}^{-1} x + \mathrm{Sin}^{-1} y = \pi - \mathrm{Sin}^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right).$$

3. *If* $-1 \leq x, y < 0$ *and* $x^2 + y^2 > 1$, *then*

$$\mathrm{Sin}^{-1} x + \mathrm{Sin}^{-1} y = -\pi - \mathrm{Sin}^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right).$$

4. *If* $-1 \le x, y \le 1$ *and* $x^2 + y^2 \le 1$, *then*

$$\text{Sin}^{-1} x - \text{Sin}^{-1} y = \text{Sin}^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right).$$

5. *If* $0 < xy \le 1$ *and* $x^2 + y^2 > 1$, *then*

$$\text{Sin}^{-1} x - \text{Sin}^{-1} y = \text{Sin}^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right).$$

6. *If* $0 < x \le 1, -1 \le y < 0$ *and* $x^2 + y^2 > 1$, *then*

$$\text{Sin}^{-1} x - \text{Sin}^{-1} y = \pi - \text{Sin}^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right).$$

7. *If* $-1 \le x < 0, 0 < y \le 1$ *and* $x^2 + y^2 > 1$, *then*

$$\text{Sin}^{-1} x - \text{Sin}^{-1} y = -\pi - \text{Sin}^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right).$$

8. *If* $-1 \le x, y \le 1$ *and* $x + y \ge 0$, *then*

$$\text{Cos}^{-1} x + \text{Cos}^{-1} y = \text{Cos}^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right).$$

9. *If* $-1 \le x, y \le 1$ *and* $x + y < 0$ *then*

$$\text{Cos}^{-1} x + \text{Cos}^{-1} y = 2\pi - \text{Cos}^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right).$$

10. *If* $-1 \le x, y \le 1$ *and* $x \le y$ *then*

$$\text{Cos}^{-1} x - \text{Cos}^{-1} y = \text{Cos}^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right).$$

11. *If* $-1 \le y \le 0, 0 < x \le 1$ *and* $x \ge y$ *then*

$$\text{Cos}^{-1} x - \text{Cos}^{-1} y = -\text{Cos}^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right).$$

Now we derive a formula for $\text{Tan}^{-1} x + \text{Tan}^{-1} y$ in the following.

8.3.11 Theorem

Suppose $x > 0$ *and* $y > 0$

(i) *If* $xy < 1$, *then* $\text{Tan}^{-1} x + \text{Tan}^{-1} y = \text{Tan}^{-1}\left(\dfrac{x+y}{1-xy}\right).$

(ii) *If* $xy > 1$, *then* $\text{Tan}^{-1} x + \text{Tan}^{-1} y = \pi + \text{Tan}^{-1}\left(\dfrac{x+y}{1-xy}\right).$

(iii) *If* $xy = 1$, *then* $\text{Tan}^{-1} x + \text{Tan}^{-1} y = \dfrac{\pi}{2}.$

(iv) $\text{Tan}^{-1} x - \text{Tan}^{-1} y = \text{Tan}^{-1} \dfrac{x-y}{1+xy}.$

Proof: Let $\alpha = \text{Tan}^{-1} x$ and $\beta = \text{Tan}^{-1} y$. Then $\tan\alpha = x$ and $\tan\beta = y$. Since x, y are positive, we get $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$. Hence $0 < \alpha + \beta < \pi$.

If $xy \neq 1$, then $\tan\alpha . \tan\beta \neq 1$, so that $\alpha + \beta \neq \frac{\pi}{2}$ and

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta} = \frac{x + y}{1 - xy} \quad \text{... (1)}$$

(i) Let $xy < 1$. Then $\tan(\alpha + \beta) = \frac{x + y}{1 - xy} > 0$, by (1). Hence $0 < \alpha + \beta < \frac{\pi}{2}$.

$$\therefore \text{ from (1)} \quad \alpha + \beta = \text{Tan}^{-1}\left(\frac{x + y}{1 - xy}\right).$$

$$\text{Therefore } \text{Tan}^{-1} x + \text{Tan}^{-1} y = \text{Tan}^{-1} \frac{x + y}{1 - xy}.$$

(ii) Let $xy > 1$. Then from (1), $\tan(\alpha + \beta) < 0$. So

$$\frac{\pi}{2} < \alpha + \beta < \pi. \quad \text{Thus} \quad \frac{-\pi}{2} < \alpha + \beta - \pi < 0 \text{ and}$$

$$\tan(\alpha + \beta - \pi) = -\tan(\pi - (\alpha + \beta)) = \tan(\alpha + \beta) = \frac{x + y}{1 - xy} \text{ from (1).}$$

$$\therefore \alpha + \beta - \pi = \text{Tan}^{-1} \frac{x + y}{1 - xy}$$

$$\text{Thus} \quad \text{Tan}^{-1} x + \text{Tan}^{-1} y = \pi + \text{Tan}^{-1} \frac{x + y}{1 - xy}.$$

(iii) Let $xy = 1$. Then $y = \frac{1}{x}$.

$$\text{Now} \quad \text{Tan}^{-1} x + \text{Tan}^{-1} y = \text{Tan}^{-1} x + \text{Tan}^{-1} \frac{1}{x} = \text{Tan}^{-1} x + \text{Cot}^{-1} x = \frac{\pi}{2}.$$

(iv) Let $x > 0$ and $y > 0$. Suppose $\text{Tan}^{-1} x = \alpha$ and $\text{Tan}^{-1} y = \beta$.

Then $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$ and $\tan\alpha = x$, $\tan\beta = y$.

Now $\alpha - \beta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha . \tan\beta} = \frac{x - y}{1 + xy}$.

Hence, $\alpha - \beta = \text{Tan}^{-1} \frac{x - y}{1 + xy}$.

Therefore, $\text{Tan}^{-1}x - \text{Tan}^{-1}y = \text{Tan}^{-1}\left(\dfrac{x-y}{1+xy}\right)$.

Similar to the results in the above Theorem 8.3.11, we also have the following. We state these results without proof.

8.3.12 Theorem

(i) *If* $x < 0,\ y < 0$ *and* $xy < 1$, *then* $\text{Tan}^{-1}x + \text{Tan}^{-1}y = \text{Tan}^{-1}\left(\dfrac{x+y}{1-xy}\right)$.

(ii) *If* $x < 0,\ y < 0$ *and* $xy > 1$, *then*

$$\text{Tan}^{-1}x + \text{Tan}^{-1}y = -\pi + \text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right).$$

(iii) *If* $xy > -1$, *then* $\text{Tan}^{-1}x - \text{Tan}^{-1}y = \text{Tan}^{-1}\left(\dfrac{x-y}{1+xy}\right)$.

(iv) *If* $x > 0,\ y < 0$ *and* $xy < -1$, *then*

$$\text{Tan}^{-1}x - \text{Tan}^{-1}y = \pi + \text{Tan}^{-1}\left(\frac{x-y}{1+xy}\right).$$

(v) *If* $x < 0,\ y > 0$ *and* $xy < -1$, *then*

$$\text{Tan}^{-1}x - \text{Tan}^{-1}y = -\pi + \text{Tan}^{-1}\left(\frac{x-y}{1+xy}\right).$$

(vi) *If* x, y, z *have same sign and* $xy + yz + zx < 1$

then $\text{Tan}^{-1}x + \text{Tan}^{-1}y + \text{Tan}^{-1}z = \text{Tan}^{-1}\left(\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right)$.

On substituting $x = y$ *in the above formulae, we get the following .*

8.3.13 Corollary

1. $$\begin{aligned} 2\,\text{Sin}^{-1}x &= \text{Sin}^{-1}\left(2x\sqrt{1-x^2}\right), \ \textit{if}\ \ x \in \left[\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] \\ &= \pi - \text{Sin}^{-1}\left(2x\sqrt{1-x^2}\right), \ \textit{if}\ \ x \in \left[\frac{1}{\sqrt{2}}, 1\right] \\ &= -\pi - \text{Sin}^{-1}\left(2x\sqrt{1-x^2}\right), \ \textit{if}\ \ x \in \left[-1, \frac{-1}{\sqrt{2}}\right]. \end{aligned}$$

2. $$\begin{aligned} 2\,\text{Cos}^{-1}x &= \text{Cos}^{-1}\left(2x^2-1\right), \ \textit{if}\ \ x \in [0,1] \\ &= 2\pi - \text{Cos}^{-1}\left(2x^2-1\right), \ \textit{if}\ \ x \in [-1,0]. \end{aligned}$$

3. $$2\,\mathrm{Tan}^{-1}x = \mathrm{Tan}^{-1}\frac{2x}{1-x^2},\ if\ x\in(-1,1)$$
$$= \pi + \mathrm{Tan}^{-1}\frac{2x}{1-x^2},\ if\ x\in(1,\infty)$$
$$= -\pi + \mathrm{Tan}^{-1}\left(\frac{2x}{1-x^2}\right),\ if\ x\in(-\infty,-1).$$

4. $$2\,\mathrm{Tan}^{-1}x = \mathrm{Sin}^{-1}\frac{2x}{1+x^2},\ if\ x\in[-1,1]$$
$$= \pi - \mathrm{Sin}^{-1}\frac{2x}{1+x^2},\ if\ x\in(1,\infty)$$
$$= -\pi - \mathrm{Sin}^{-1}\frac{2x}{1+x^2},\ if\ x\in(-\infty,-1).$$

5. $$2\,\mathrm{Tan}^{-1}x = \mathrm{Cos}^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ if\ x\in[0,\infty)$$
$$= -\mathrm{Cos}^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ if\ x\in(-\infty,0].$$

6. $$3\,\mathrm{Sin}^{-1}x = \mathrm{Sin}^{-1}(3x-4x^3),\ if\ x\in\left[\frac{-1}{2},\frac{1}{2}\right]$$
$$= \pi - \mathrm{Sin}^{-1}(3x-4x^3),\ if\ x\in\left(\frac{1}{2},1\right]$$
$$= -\pi - \mathrm{Sin}^{-1}(3x-4x^3),\ if\ x\in\left[-1,\frac{-1}{2}\right).$$

7. $$3\,\mathrm{Cos}^{-1}x = \mathrm{Cos}^{-1}(4x^3-3x),\ if\ x\in\left[\frac{1}{2},1\right]$$
$$= 2\pi - \mathrm{Cos}^{-1}(4x^3-3x),\ if\ x\in\left(-\frac{1}{2},\frac{1}{2}\right]$$
$$= 2\pi + \mathrm{Cos}^{-1}(4x^3-3x),\ if\ x\in\left[-1,\frac{-1}{2}\right].$$

8. $$3\,\mathrm{Tan}^{-1}x = \mathrm{Tan}^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),\ if\ x\in\left(\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$
$$= \pi + \mathrm{Tan}^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),\ if\ x\in\left(\frac{1}{\sqrt{3}},\infty\right)$$
$$= -\pi + \mathrm{Tan}^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),\ if\ x\in\left(-\infty,-\frac{1}{\sqrt{3}}\right).$$

8.3.14 Solved Problems

1. Problem: *Find the values of the following.*

(i) $\mathrm{Sin}^{-1}\left(-\frac{1}{2}\right)$ (ii) $\mathrm{Cos}^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ (iii) $\mathrm{Tan}^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(iv) $\mathrm{Cot}^{-1}(-1)$ (v) $\mathrm{Sec}^{-1}(-\sqrt{2})$ (vi) $\mathrm{Cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Solution

(i) $\sin\left(-\frac{\pi}{6}\right) = \frac{-1}{2}$ and $\frac{-\pi}{6} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \mathrm{Sin}^{-1}\left(\frac{-1}{2}\right) = \frac{-\pi}{6}$.

(ii) $\cos\left(\pi - \frac{\pi}{6}\right) = -\cos\frac{\pi}{6} = \frac{-\sqrt{3}}{2}$ and

$$\frac{5\pi}{6} \in [0, \pi] \Rightarrow \mathrm{Cos}^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \frac{5\pi}{6}.$$

(iii) $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ and $\frac{\pi}{6} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \mathrm{Tan}^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$.

(iv) $\cot\frac{3\pi}{4} = \cot\left(\pi - \frac{\pi}{4}\right) = -\cot\frac{\pi}{4} = -1$ and

$$\frac{3\pi}{4} \in (0, \pi) \Rightarrow \mathrm{Cot}^{-1}(-1) = \frac{3\pi}{4}.$$

(v) $\sec\frac{3\pi}{4} = \sec\left(\pi - \frac{\pi}{4}\right) = -\sec\frac{\pi}{4} = -\sqrt{2}$ and

$$\frac{3\pi}{4} \in \left(\frac{\pi}{2}, \pi\right] \Rightarrow \mathrm{Sec}^{-1}(-\sqrt{2}) = \frac{3\pi}{4}.$$

(vi) $\mathrm{cosec}\frac{\pi}{3} = \frac{2}{\sqrt{3}}$ and $\frac{\pi}{3} \in \left(0, \frac{\pi}{2}\right] \Rightarrow \mathrm{Cosec}^{-1}\frac{2}{\sqrt{3}} = \frac{\pi}{3}$.

2. Problem: *Find the values of the following .*

(i) $\mathrm{Sin}^{-1}\left(\sin\frac{4\pi}{3}\right)$ (ii) $\mathrm{Cos}^{-1}\left(\cos\frac{4\pi}{3}\right)$ (iii) $\mathrm{Tan}^{-1}\left(\tan\frac{4\pi}{3}\right)$

Solution

(i) $\mathrm{Sin}^{-1}\left(\sin\frac{4\pi}{3}\right) = \mathrm{Sin}^{-1}\left(\sin\left(\pi + \frac{\pi}{3}\right)\right) = \mathrm{Sin}^{-1}\left(-\sin\frac{\pi}{3}\right)$

$$= \mathrm{Sin}^{-1}\left(\sin\left(\frac{-\pi}{3}\right)\right) = \frac{-\pi}{3}, \text{ since } \frac{-\pi}{3} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right].$$

(ii) $\text{Cos}^{-1}\left(\cos\frac{4\pi}{3}\right) = \text{Cos}^{-1}\left(\cos\left(\pi+\frac{\pi}{3}\right)\right) = \text{Cos}^{-1}\left(-\cos\frac{\pi}{3}\right)$

$$= \text{Cos}^{-1}\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}, \text{ since } \frac{2\pi}{3} \in (0,\pi).$$

(iii) $\text{Tan}^{-1}\left(\tan\frac{4\pi}{3}\right) = \text{Tan}^{-1}\left(\tan\left(\pi+\frac{\pi}{3}\right)\right) = \text{Tan}^{-1}\left(\tan\frac{\pi}{3}\right)$

$$= \frac{\pi}{3}, \text{ since } \frac{\pi}{3} \in \left(\frac{-\pi}{2},\frac{\pi}{2}\right).$$

3. Problem: *Find the values of the following.*

(i) $\sin\left(\text{Cos}^{-1}\frac{5}{13}\right)$ (ii) $\tan\left(\text{Sec}^{-1}\frac{25}{7}\right)$ (iii) $\cos\left(\text{Tan}^{-1}\frac{24}{7}\right)$

Solution:

(i) $\sin\left(\text{Cos}^{-1}\frac{5}{13}\right) = \sin\left(\text{Sin}^{-1}\frac{12}{13}\right) = \frac{12}{13}$.

(ii) $\tan\left(\text{Sec}^{-1}\frac{25}{7}\right) = \tan\left(\text{Tan}^{-1}\frac{24}{7}\right) = \frac{24}{7}$.

(iii) $\cos\left(\text{Tan}^{-1}\frac{24}{7}\right) = \cos\left(\text{Cos}^{-1}\frac{7}{25}\right) = \frac{7}{25}$.

4. Problem: *Find the values of the following.*

(i) $\sin^2\left(\text{Tan}^{-1}\frac{3}{4}\right)$ (ii) $\sin\left(\frac{\pi}{2} - \text{Sin}^{-1}\left(-\frac{4}{5}\right)\right)$

(iii) $\cos\left(\text{Cos}^{-1}\left(-\frac{2}{3}\right) - \text{Sin}^{-1}\left(\frac{2}{3}\right)\right)$ (iv) $\sec^2\left(\text{Cot}^{-1}3\right) + \text{cosec}^2\left(\text{Tan}^{-1}2\right)$

Solution

(i) $\sin\left(\text{Tan}^{-1}\frac{3}{4}\right) = \sin\left(\text{Sin}^{-1}\frac{3}{5}\right) = \frac{3}{5}$

Therefore, $\sin^2\left(\text{Tan}^{-1}\frac{3}{4}\right) = \left[\sin\left(\text{Tan}^{-1}\frac{3}{4}\right)\right]^2 = \frac{9}{25}$.

(ii) $\sin\left(\frac{\pi}{2} - \text{Sin}^{-1}\left(\frac{-4}{5}\right)\right) = \sin\left(\frac{\pi}{2} + \text{Sin}^{-1}\left(\frac{4}{5}\right)\right)$ (since $\text{Sin}^{-1}(-x) = -\text{Sin}^{-1}x$)

$$= \cos\left(\text{Sin}^{-1}\frac{4}{5}\right) = \cos\left(\text{Cos}^{-1}\frac{3}{5}\right) = \frac{3}{5}.$$

(iii) $\left(\text{Cos}^{-1}\left(\frac{-2}{3}\right) - \text{Sin}^{-1}\left(\frac{2}{3}\right)\right) = \left(\pi - \text{Cos}^{-1}\frac{2}{3}\right) - \text{Sin}^{-1}\frac{2}{3}$

(since $\text{Cos}^{-1}(-x) = \pi - \text{Cos}^{-1}x$)

$$= \pi - \left(\text{Cos}^{-1}\frac{2}{3} + \text{Sin}^{-1}\frac{2}{3}\right)$$

$$= \pi - \frac{\pi}{2} = \frac{\pi}{2}.$$

Hence $\cos\left(\text{Cos}^{-1}\left(\frac{-2}{3}\right) - \text{Sin}^{-1}\frac{2}{3}\right) = \cos\frac{\pi}{2} = 0.$

(iv) If $\text{Cot}^{-1}3 = \theta$, then show that $\cot\theta = 3$.

Therefore $\sec^2\theta = 1 + \tan^2\theta = 1 + \left(\frac{1}{3}\right)^2 = \frac{10}{9}$.

That is, $\sec^2\left(\text{Cot}^{-1}(3)\right) = \frac{10}{9}$.

Again, if $\text{Tan}^{-1}2 = \alpha$, then $\tan\alpha = 2$.

Therefore, $\text{cosec}^2\alpha = 1 + \cot^2\alpha = 1 + \frac{1}{4} = \frac{5}{4}$.

Thus, $\text{cosec}^2\left(\text{Tan}^{-1}2\right) = \frac{5}{4}$.

Hence $\sec^2\left(\text{Cot}^{-1}3\right) + \text{cosec}^2\left(\text{Tan}^{-1}2\right) = \frac{10}{9} + \frac{5}{4} = \frac{85}{36}$.

5. Problem: *Find the value of* $\text{Cot}^{-1}\frac{1}{2} + \text{Cot}^{-1}\left(\frac{1}{3}\right)$.

Solution: We know that $\text{Cot}^{-1}\frac{1}{2} = \text{Tan}^{-1}2$ and $\text{Cot}^{-1}\left(\frac{1}{3}\right) = \text{Tan}^{-1}3$.

By 8.3.11, (ii)

$$\text{Cot}^{-1}\frac{1}{2} + \text{Cot}^{-1}\frac{1}{3} = \text{Tan}^{-1}2 + \text{Tan}^{-1}3 = \pi + \text{Tan}^{-1}\left(\frac{2+3}{1-6}\right)$$

$$= \pi + \text{Tan}^{-1}(-1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

6. Problem: *Prove that* $\text{Sin}^{-1}\frac{4}{5} + \text{Sin}^{-1}\frac{7}{25} = \text{Sin}^{-1}\frac{117}{125}$.

Solution

Method (i):

Let $\text{Sin}^{-1}\frac{4}{5} = \alpha$ and $\text{Sin}^{-1}\frac{7}{25} = \beta$.

Then $\sin\alpha = \frac{4}{5}$ and $\sin\beta = \frac{7}{25}$ and $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$

so that $\cos\alpha = \frac{3}{5}$, $\cos\beta = \frac{24}{25}$ and $\alpha+\beta \in (0, \pi)$

Now $\cos(\alpha+\beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

$$= \frac{3}{5}\cdot\frac{24}{25} - \frac{4}{5}\cdot\frac{7}{5} = \frac{44}{125} > 0.$$

Hence $\alpha+\beta \in \left(0, \frac{\pi}{2}\right)$. Now,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \frac{4}{5}\cdot\frac{24}{25} + \frac{3}{5}\cdot\frac{7}{25}$$

$$= \frac{96+21}{125} = \frac{117}{125}$$

$$\therefore \alpha+\beta = \text{Sin}^{-1}\left(\frac{117}{125}\right).$$

Hence, $\text{Sin}^{-1}\frac{4}{5} + \text{Sin}^{-1}\frac{7}{25} = \text{Sin}^{-1}\frac{117}{125}$.

Method (ii)

We know that $\text{Sin}^{-1}x + \text{Sin}^{-1}y = \text{Sin}^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$.

if $x > 0,\ y > 0$ and $x^2 + y^2 < 1$

Therefore, $\text{Sin}^{-1}\frac{4}{5} + \text{Sin}^{-1}\frac{7}{25} = \text{Sin}^{-1}\left\{\frac{4}{5}\sqrt{1-\frac{49}{625}} + \frac{7}{25}\sqrt{1-\frac{16}{25}}\right\}$

$$= \text{Sin}^{-1}\left(\frac{4}{5}\cdot\frac{24}{25} + \frac{7}{25}\cdot\frac{3}{5}\right) = \text{Sin}^{-1}\left(\frac{117}{125}\right).$$

7. Problem: *If* $x \in (-1, 1)$, *prove that* $2\,\text{Tan}^{-1}x = \text{Tan}^{-1}\frac{2x}{1-x^2}$.

Solution: Let $x \in (-1, 1)$ and $\text{Tan}^{-1}x = \alpha$. Then $\tan\alpha = x$ and $-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$. Now

$$\text{Tan}^{-1}\left(\frac{2x}{1-x^2}\right) = \text{Tan}^{-1}\left(\frac{2\tan\alpha}{1-\tan^2\alpha}\right) = \text{Tan}^{-1}(\tan 2\alpha) = 2\alpha, \text{ since } 2\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\therefore \text{Tan}^{-1}\frac{2x}{1-x^2} = 2\,\text{Tan}^{-1}x.$$

8. Problem: *Prove that* $\mathrm{Sin}^{-1}\frac{4}{5} + \mathrm{Sin}^{-1}\frac{5}{13} + \mathrm{Sin}^{-1}\left(\frac{16}{65}\right) = \frac{\pi}{2}$.

Solution: Let $\mathrm{Sin}^{-1}\frac{4}{5} = A$ and $\mathrm{Sin}^{-1}\frac{5}{13} = B$. Then A, B are acute angles

and $\sin A = \frac{4}{5}$, $\sin B = \frac{5}{13}$. Hence $\cos A = \frac{3}{5}$, $\cos B = \frac{12}{13}$. Now,

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$= \frac{3}{5}\cdot\frac{12}{13} - \frac{4}{5}\cdot\frac{5}{13} = \frac{16}{65}.$$

$$\Rightarrow A + B = \mathrm{Cos}^{-1}\frac{16}{65} \Rightarrow \mathrm{Sin}^{-1}\frac{4}{5} + \mathrm{Sin}^{-1}\frac{5}{13} = \mathrm{Cos}^{-1}\frac{16}{65} = \frac{\pi}{2} - \mathrm{Sin}^{-1}\left(\frac{16}{65}\right).$$

$$\Rightarrow \mathrm{Sin}^{-1}\frac{4}{5} + \mathrm{Sin}^{-1}\frac{5}{13} + \mathrm{Sin}^{-1}\frac{16}{65} = \frac{\pi}{2}.$$

9. Problem: *Prove that* $\mathrm{Cot}^{-1}9 + \mathrm{Cosec}^{-1}\frac{\sqrt{41}}{4} = \frac{\pi}{4}$.

Solution: Write $\mathrm{Cot}^{-1}9 = \alpha$ and $\mathrm{Cosec}^{-1}\frac{\sqrt{41}}{4} = \beta$.

Then $\cot\alpha = 9$ and $\mathrm{cosec}\,\beta = \frac{\sqrt{41}}{4}$.

$\Rightarrow 0 < \alpha, \beta < \frac{\pi}{2}$ and $\tan\alpha = \frac{1}{9}$, $\tan\beta = \frac{4}{5}$. Now

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{\frac{1}{9} + \frac{4}{5}}{1 - \frac{1}{9}\cdot\frac{4}{5}} = \frac{\frac{41}{45}}{\frac{41}{45}} = 1.$$

Since $\frac{1}{9}\cdot\frac{4}{5} = \frac{4}{45} < 1$, we get $\alpha + \beta = \frac{\pi}{4}$.

Hence $\mathrm{Cot}^{-1}9 + \mathrm{Cosec}^{-1}\frac{\sqrt{41}}{4} = \frac{\pi}{4}$.

10. Problem: *Show that* $\cot\left(\mathrm{Sin}^{-1}\sqrt{\frac{13}{17}}\right) = \sin\left(\mathrm{Tan}^{-1}\frac{2}{3}\right)$.

Solution: Let $\text{Sin}^{-1}\sqrt{\frac{13}{17}} = \alpha$. Then $\sin\alpha = \sqrt{\frac{13}{17}}$

Hence $\cot\alpha = \frac{2}{\sqrt{13}}$. That is $\cot\left(\text{Sin}^{-1}\sqrt{\frac{13}{17}}\right) = \frac{2}{\sqrt{13}}\cdot$

Suppose $\text{Tan}^{-1}\frac{2}{3} = \beta$. That is $\tan\beta = \frac{2}{3}\cdot$

So that $\sin\beta = \frac{2}{\sqrt{13}}$. That is $\sin\left(\text{Tan}^{-1}\frac{2}{3}\right) = \frac{2}{\sqrt{13}}\cdot$

Hence $\cot\left(\text{Sin}^{-1}\sqrt{\frac{13}{17}}\right) = \sin\left(\text{Tan}^{-1}\frac{2}{3}\right)$.

11. Problem: *Find the value of* $\tan\left(2\,\text{Tan}^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right)$.

Solution: Let $\text{Tan}^{-1}\frac{1}{5} = \alpha$. Then $0 < \alpha < \frac{\pi}{2}$ and $\tan\alpha = \frac{1}{5}\cdot$

So that $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\times\frac{1}{5}}{1-\frac{1}{25}} = \frac{2}{5}\times\frac{25}{24} = \frac{5}{12}\cdot$

Now, $\tan\left(2\,\text{Tan}^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right)$

$$= \tan\left(2\alpha - \frac{\pi}{4}\right) = \frac{\tan 2\alpha - \tan\frac{\pi}{4}}{1 + \tan 2\alpha\,.\,\tan\frac{\pi}{4}}$$

$$= \frac{\frac{5}{12} - 1}{1 + \frac{5}{12}.1} = \frac{-7}{12} \times \frac{12}{17} = \frac{-7}{17}.$$

12. Problem: *Prove that* $\text{Sin}^{-1}\frac{4}{5} + 2\,\text{Tan}^{-1}\frac{1}{3} = \frac{\pi}{2}$.

Solution: Let $\text{Tan}^{-1}\frac{1}{3} = \beta$. Then $0 < \beta < \frac{\pi}{2}$, $\tan\beta = \frac{1}{3}$.

Now $\tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta} = \frac{2\,.\,\frac{1}{3}}{1-\frac{1}{9}} = \frac{2}{3}\times\frac{9}{8} = \frac{3}{4}$. Thus $0 < 2\beta < \frac{\pi}{2}$

$\Rightarrow \cos 2\beta = \frac{4}{5}\cdot$

Thus $2\beta = \mathrm{Cos}^{-1}\frac{4}{5}\cdot$

Now $\mathrm{Sin}^{-1}\frac{4}{5} + 2\,\mathrm{Tan}^{-1}\frac{1}{3} = \mathrm{Sin}^{-1}\frac{4}{5} + \mathrm{Cos}^{-1}\frac{4}{5} = \frac{\pi}{2}$.

13. Problem: *Prove that* $\cos\left(2\,\mathrm{Tan}^{-1}\frac{1}{7}\right) = \sin\left(4\,\mathrm{Tan}^{-1}\frac{1}{3}\right)$.

Solution: Let $\mathrm{Tan}^{-1}\frac{1}{7} = \alpha$ and $\mathrm{Tan}^{-1}\frac{1}{3} = \beta\cdot$

Then $\tan\alpha = \frac{1}{7}$ and $\tan\beta = \frac{1}{3}$. Also $0 < \alpha, \beta < \frac{\pi}{2}$

$$\text{Now,}\quad \cos\left(2\,\mathrm{Tan}^{-1}\frac{1}{7}\right) = \cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha} = \frac{1-\frac{1}{49}}{1+\frac{1}{49}} = \frac{48}{50} = \frac{24}{25} \qquad \text{... (1)}$$

$$\tan\beta = \frac{1}{3} \Rightarrow \tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta} = \frac{2\cdot\frac{1}{3}}{1-\frac{1}{9}} = \frac{2}{3}\times\frac{9}{8} = \frac{3}{4}\cdot$$

$$\text{Now,}\quad \sin\left(4\,\mathrm{Tan}^{-1}\frac{1}{3}\right) = \sin(4\beta) = \sin(2(2\beta))$$

$$= \frac{2\tan 2\beta}{1+\tan^2 2\beta} = \frac{2\cdot\frac{3}{4}}{1+\frac{9}{16}} = \frac{3}{2}\times\frac{16}{25} = \frac{24}{25} \qquad \text{... (2)}$$

Therefore, from (1) and (2), we get $\cos\left(2\,\mathrm{Tan}^{-1}\frac{1}{7}\right) = \sin\left(4\,\mathrm{Tan}^{-1}\frac{1}{3}\right)$.

14. Problem: *If* $\mathrm{Sin}^{-1}x + \mathrm{Sin}^{-1}y + \mathrm{Sin}^{-1}z = \pi$, *then prove that*

$$x^4 + y^4 + z^4 + 4x^2y^2z^2 = 2\left(x^2y^2 + y^2z^2 + z^2x^2\right).$$

Solution: Let $\mathrm{Sin}^{-1}x = \alpha$, $\mathrm{Sin}^{-1}y = \beta$ and $\mathrm{Sin}^{-1}z = \gamma$. Then

$$\sin\alpha = x,\ \sin\beta = y \text{ and } \sin\gamma = z \text{ and } \alpha+\beta+\gamma = \pi \quad \text{(given)}$$

Now $\alpha + \beta = \pi - \gamma$

$\Rightarrow \cos(\alpha+\beta) = \cos(\pi-\gamma) \Rightarrow \cos\alpha\cos\beta - \sin\alpha\sin\beta = -\cos\gamma$

$\Rightarrow \sqrt{1-x^2}\sqrt{1-y^2} - xy = -\sqrt{1-z^2}$

$\Rightarrow \sqrt{1-x^2}\sqrt{1-y^2} = xy - \sqrt{1-z^2}$.

On squaring both sides we get

$$\left(1-x^2\right)\left(1-y^2\right)=x^2y^2+1-z^2-2xy\sqrt{1-z^2}$$

$$\Rightarrow \quad 1-x^2-y^2+x^2y^2=x^2y^2+1-z^2-2xy\sqrt{1-z^2}$$

$$\Rightarrow \quad 2xy\sqrt{1-z^2}=x^2+y^2-z^2.$$

Again on squaring both sides, we get

$$4x^2y^2\left(1-z^2\right)=\left(x^2+y^2-z^2\right)^2$$

$$\Rightarrow \quad 4x^2y^2-4x^2y^2z^2 = x^4+y^4+z^4+2x^2y^2-2y^2z^2-2x^2z^2$$

$$\Rightarrow \quad x^4+y^4+z^4+4x^2y^2z^2=2x^2y^2+2y^2z^2+2x^2z^2.$$

15. Problem: *If* $\text{Cos}^{-1}\dfrac{p}{a}+\text{Cos}^{-1}\dfrac{q}{b}=\alpha$ *, then prove that*

$$\frac{p^2}{a^2}-\frac{2pq}{ab}\cdot\cos\alpha+\frac{q^2}{b^2}=\sin^2\alpha.$$

Solution: Let $\text{Cos}^{-1}\dfrac{p}{a}=\text{A}$ and $\text{Cos}^{-1}\dfrac{q}{b}=\text{B}$.

Then $\cos \text{A}=\dfrac{p}{a}$, $\cos \text{B}=\dfrac{q}{b}$ and $\text{A}+\text{B}=\alpha$ (given)

Now, $\cos\alpha=\cos(\text{A}+\text{B})=\cos \text{A}\cos \text{B}-\sin \text{A}\sin \text{B}$

$$=\frac{p}{a}\cdot\frac{q}{b}-\sqrt{1-\frac{p^2}{a^2}}\cdot\sqrt{1-\frac{q^2}{b^2}}$$

$$\Rightarrow \quad \sqrt{1-\frac{p^2}{a^2}}\cdot\sqrt{1-\frac{q^2}{b^2}}=\frac{pq}{ab}-\cos\alpha$$

On squaring both sides, we get

$$\left(1-\frac{p^2}{a^2}\right)\cdot\left(1-\frac{q^2}{b^2}\right)=\left(\frac{pq}{ab}-\cos\alpha\right)^2$$

$$\Rightarrow \quad 1-\frac{p^2}{a^2}-\frac{q^2}{b^2}+\frac{p^2q^2}{a^2b^2}=\frac{p^2q^2}{a^2b^2}-\frac{2pq}{ab}\cos\alpha+\cos^2\alpha$$

$$\Rightarrow \quad \frac{p^2}{a^2}-\frac{2pq}{ab}\cos\alpha+\frac{q^2}{b^2}=1-\cos^2\alpha=\sin^2\alpha.$$

16. Problem: *Solve* $\text{arc sin}\left(\frac{5}{x}\right) + \text{arc sin}\frac{12}{x} = \frac{\pi}{2}$ $(x > 0)$.

Solution: Given that $\text{Sin}^{-1}\frac{5}{x} + \text{Sin}^{-1}\frac{12}{x} = \frac{\pi}{2}$ and $x > 0$

$$\Rightarrow \text{Sin}^{-1}\frac{5}{x} = \frac{\pi}{2} - \text{Sin}^{-1}\frac{12}{x} = \text{Cos}^{-1}\frac{12}{x} = \text{Sin}^{-1}\sqrt{1-\frac{144}{x^2}}$$

$$\Rightarrow \frac{5}{x} = \sqrt{1-\frac{144}{x^2}}$$

On squaring both sides we get

$$\frac{25}{x^2} = 1 - \frac{144}{x^2} \Rightarrow x^2 = 169 \Rightarrow x = \pm 13 \Rightarrow x = 13 \text{ (since } x > 0\text{)}.$$

17. Problem: *Solve* $\text{Sin}^{-1}\frac{3x}{5} + \text{Sin}^{-1}\frac{4x}{5} = \text{Sin}^{-1}x$.

Solution: $\text{Sin}^{-1}\frac{3x}{5} + \text{Sin}^{-1}\frac{4x}{5} = \text{Sin}^{-1}x$

$$\Rightarrow x = \sin\left(\text{Sin}^{-1}\frac{3x}{5} + \text{Sin}^{-1}\frac{4x}{5}\right)$$

$$= \frac{3x}{5}\sqrt{1-\frac{16x^2}{25}} + \frac{4x}{5}\sqrt{1-\frac{9x^2}{25}}$$

$$\Rightarrow x = 0 \text{ or } 25 = 3\sqrt{25-16x^2} + 4\sqrt{25-9x^2}$$

$$x \neq 0 \Rightarrow 4\sqrt{25-9x^2} = 25 - 3\sqrt{25-16x^2}.$$

On squaring both sides, we get

$$16\left(25-9x^2\right) = 625 - 150\sqrt{25-16x^2} + 9\left(25-16x^2\right)$$

$$\Rightarrow 400 - 144x^2 = 625 - 150\sqrt{25-16x^2} + 225 - 144x^2$$

$$\Rightarrow 150\sqrt{25-16x^2} = 450$$

$$\Rightarrow \sqrt{25-16x^2} = 3 \Rightarrow 25 - 16x^2 = 9$$

$$\Rightarrow 16x^2 = 16 \Rightarrow x = \pm 1.$$

Thus, we get $x = -1, 0, 1$. We can verify that all these values of x satisfy the given equation.

18. Problem: *Solve* $\text{Sin}^{-1} x + \text{Sin}^{-1} 2x = \frac{\pi}{3}$.

Solution: Given that $\text{Sin}^{-1} x + \text{Sin}^{-1} 2x = \frac{\pi}{3}$

$$\Rightarrow \cos\left(\text{Sin}^{-1} x + \text{Sin}^{-1} 2x\right) = \cos\frac{\pi}{3}$$

$$\Rightarrow \sqrt{1-x^2}\,\sqrt{1-4x^2} - x.2x = \frac{1}{2}$$

$$\Rightarrow \sqrt{1-x^2}\,\sqrt{1-4x^2} = 2x^2 + \frac{1}{2}$$

On squaring both sides, we get

$$\left(1-x^2\right)\left(1-4x^2\right) = \left(2x^2 + \frac{1}{2}\right)^2$$

$$\Rightarrow 1 - 5x^2 + 4x^4 = 4x^4 + 2x^2 + \frac{1}{4}$$

$$\Rightarrow 7x^2 = \frac{3}{4} \Rightarrow x^2 = \frac{3}{28} \Rightarrow x = \pm\frac{\sqrt{3}}{2\sqrt{7}}.$$

But $x = -\frac{\sqrt{3}}{2\sqrt{7}}$ does not satisfy the given equation since $\text{Sin}^{-1} x$ and $\text{Sin}^{-1} 2x$ both are negative in this case.

$$\therefore \text{ Only solution is } x = \frac{\sqrt{3}}{2\sqrt{7}}.$$

19. Problem: *If* $\sin\left[2\,\text{Cos}^{-1}\left\{\cot\left(2\,\text{Tan}^{-1} x\right)\right\}\right] = 0$, *find x.*

Solution: $\sin\left[2\,\text{Cos}^{-1}\left\{\cot\left(2\,\text{Tan}^{-1} x\right)\right\}\right] = 0$

$$\Leftrightarrow 2\,\text{Cos}^{-1}\left\{\cot\left(2\,\text{Tan}^{-1} x\right)\right\} = 0 \text{ or } \pi \text{ or } 2\pi$$

(Since the range of Cos^{-1} is $[0, \pi]$)

$$\Leftrightarrow \text{Cos}^{-1}\left\{\cot\left(2\,\text{Tan}^{-1} x\right)\right\} = 0 \text{ or } \frac{\pi}{2} \text{ or } \pi$$

$$\Leftrightarrow \cot\left(2\,\text{Tan}^{-1} x\right) = 1 \text{ or } 0 \text{ or } -1$$

$$\Leftrightarrow 2\,\text{Tan}^{-1} x = \pm\frac{\pi}{4} \text{ or } \pm\frac{\pi}{2} \text{ or } \pm\frac{3\pi}{4}$$

$$\Leftrightarrow \text{Tan}^{-1} x = \pm\frac{\pi}{8} \text{ or } \pm\frac{\pi}{4} \text{ or } \pm\frac{3\pi}{8}$$

$$\Leftrightarrow x = \pm\left(\sqrt{2}-1\right) \text{ or } \pm 1 \text{ or } \pm\left(\sqrt{2}+1\right).$$

20. Problem: *Prove that* $\cos\left[\mathrm{Tan}^{-1}\left\{\sin\left(\mathrm{Cot}^{-1}x\right)\right\}\right] = \sqrt{\dfrac{x^2+1}{x^2+2}}$.

Solution: Let $\mathrm{Cot}^{-1}x = \theta$. Then $\cot\theta = x$ and $0 < \theta < \pi$.

$$\sin\left(\mathrm{Cot}^{-1}x\right) = \sin\theta = \frac{1}{\operatorname{cosec}\theta} = \frac{1}{\sqrt{1+\cot^2\theta}} = \frac{1}{\sqrt{1+x^2}} \text{ (since } 0<\theta<\pi)$$

Now $\mathrm{Tan}^{-1}\left(\sin\left(\mathrm{Cot}^{-1}x\right)\right) = \mathrm{Tan}^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right) = \alpha$ (say)

Then $\tan\alpha = \dfrac{1}{\sqrt{1+x^2}}$ and $0 < \alpha < \dfrac{\pi}{2}$

$$\cos\left[\mathrm{Tan}^{-1}\left\{\sin\left(\mathrm{Cot}^{-1}x\right)\right\}\right] = \cos\alpha = \frac{1}{\sec\alpha} = \frac{1}{\sqrt{1+\tan^2\alpha}}$$

$$= \frac{1}{\sqrt{1+\dfrac{1}{1+x^2}}} = \sqrt{\frac{1+x^2}{2+x^2}}.$$

Exercise 8(a)

I. 1. Evaluate the following.

(i) $\mathrm{Sin}^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

(ii) $\mathrm{Cos}^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$

(iii) $\mathrm{Sec}^{-1}\left(-\sqrt{2}\right)$

(iv) $\mathrm{Cot}^{-1}\left(-\sqrt{3}\right)$

(v) $\mathrm{Sin}\left[\dfrac{\pi}{3} - \mathrm{Sin}^{-1}\left(-\dfrac{1}{2}\right)\right]$

(vi) $\mathrm{Sin}^{-1}\left[\mathrm{Sin}\left(\dfrac{5\pi}{6}\right)\right]$

(vii) $\mathrm{Cos}^{-1}\left[\mathrm{Cos}\dfrac{5\pi}{4}\right]$

2. Find the values of

(i) $\sin\left(\mathrm{Cos}^{-1}\dfrac{3}{5}\right)$

(ii) $\tan\left(\operatorname{cosec}^{-1}\dfrac{65}{63}\right)$

(iii) $\sin\left(2\mathrm{Sin}^{-1}\dfrac{4}{5}\right)$ (iv) $\mathrm{Sin}^{-1}\left(\sin\dfrac{33\pi}{7}\right)$

(v) $\mathrm{Cos}^{-1}\left(\cos\dfrac{17\pi}{6}\right)$

3. **Simplify each of the following.**

(i) $\mathrm{Tan}^{-1}\left(\dfrac{\sin x}{1+\cos x}\right)$ (ii) $\mathrm{Tan}^{-1}(\sec x+\tan x)$

(iii) $\mathrm{Tan}^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}$

(iv) $\mathrm{Sin}^{-1}(2\cos^2\theta-1)+\mathrm{Cos}^{-1}(1-2\sin^2\theta)$

(v) $\tan^{-1}\left[x+\sqrt{1+x^2}\right];\ x\in\mathbf{R}$

II. 1. Prove that

(i) $\mathrm{Sin}^{-1}\dfrac{3}{5}+\mathrm{Sin}^{-1}\dfrac{8}{17}=\mathrm{Cos}^{-1}\dfrac{36}{85}$

(ii) $\mathrm{Sin}^{-1}\dfrac{3}{5}+\mathrm{Cos}^{-1}\dfrac{12}{13}=\mathrm{Cos}^{-1}\dfrac{33}{65}$

(iii) $\tan\left\{\mathrm{Cot}^{-1}9+\mathrm{Cosec}^{-1}\dfrac{\sqrt{41}}{4}\right\}=1$

(iv) $\mathrm{Cos}^{-1}\dfrac{4}{5}+\mathrm{Sin}^{-1}\dfrac{3}{\sqrt{34}}=\mathrm{Tan}^{-1}\dfrac{27}{11}$

2. **Find the values of**

(i) $\sin\left(\mathrm{Cos}^{-1}\dfrac{3}{5}+\mathrm{Cos}^{-1}\dfrac{12}{13}\right)$ (ii) $\tan\left(\mathrm{Sin}^{-1}\dfrac{3}{5}+\mathrm{Cos}^{-1}\dfrac{5}{\sqrt{34}}\right)$

(iii) $\cos\left(\mathrm{Sin}^{-1}\dfrac{3}{5}+\mathrm{Sin}^{-1}\dfrac{5}{13}\right)$

3. **Prove that**

(i) $\cos\left(2\mathrm{Tan}^{-1}\dfrac{1}{7}\right)=\sin\left(2\mathrm{Tan}^{-1}\dfrac{3}{4}\right)$

(ii) $\tan\left[2\mathrm{Tan}^{-1}\left(\dfrac{\sqrt{5}-1}{2}\right)\right]=2$

(iii) $\cos\left\{2\left[\mathrm{Tan}^{-1}\dfrac{1}{4}+\mathrm{Tan}^{-1}\dfrac{2}{9}\right]\right\}=\dfrac{3}{5}$

4. Prove that

(i) $\text{Tan}^{-1}\frac{1}{7}+\text{Tan}^{-1}\frac{1}{13}-\text{Tan}^{-1}\frac{2}{9}=0$

(ii) $\text{Tan}^{-1}\frac{1}{2}+\text{Tan}^{-1}\frac{1}{5}+\text{Tan}^{-1}\frac{1}{8}=\frac{\pi}{4}$

(iii) $\text{Tan}^{-1}\frac{3}{4}+\text{Tan}^{-1}\frac{3}{5}-\text{Tan}^{-1}\frac{8}{19}=\frac{\pi}{4}$

(iv) $\text{Tan}^{-1}\frac{1}{7}+\text{Tan}^{-1}\frac{1}{8}=\text{Cot}^{-1}\frac{201}{43}+\text{Cot}^{-1}18$

5. (i) Show that $\sec^2(\text{Tan}^{-1}2)+\text{cosec}^2(\text{Cot}^{-1}2)=10$.

(ii) Find the value of $\tan\left(\text{Cos}^{-1}\frac{4}{5}+\text{Tan}^{-1}\frac{2}{3}\right)$.

(iii) If $\text{Sin}^{-1}x-\text{Cos}^{-1}x=\frac{\pi}{6}$, then find x.

III. 1. Prove that

(i) $2\,\text{Sin}^{-1}\frac{3}{5}-\text{Cos}^{-1}\frac{5}{13}=\text{Cos}^{-1}\frac{323}{325}$

(ii) $\text{Sin}^{-1}\frac{4}{5}+2\,\text{Tan}^{-1}\frac{1}{3}=\frac{\pi}{2}$

(iii) $4\,\text{Tan}^{-1}\frac{1}{5}+\text{Tan}^{-1}\frac{1}{99}-\text{Tan}^{-1}\frac{1}{70}=\frac{\pi}{4}$

2. (i) If $\alpha=\text{Tan}^{-1}\left[\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right]$, then prove that $x^2=\sin 2\alpha$.

(ii) Prove that $\tan\left\{2\text{Tan}^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)\right\}=x$.

(iii) Prove that $\sin\left[\text{Cot}^{-1}\frac{2x}{1-x^2}+\text{Cos}^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]=1$.

(iv) Prove that $\tan\left\{\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right\}+\tan\left\{\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right\}=\frac{2b}{a}$.

3. (i) If $\text{Cos}^{-1}p+\text{Cos}^{-1}q+\text{Cos}^{-1}r=\pi$, then prove that $p^2+q^2+r^2+2pqr=1$.

(ii) If $\text{Sin}^{-1}\left(\frac{2p}{1+p^2}\right) - \text{Cos}^{-1}\left(\frac{1-q^2}{1+q^2}\right) = \text{Tan}^{-1}\left(\frac{2x}{1-x^2}\right)$ then prove that

$x = \frac{p-q}{1+pq}$.

(iii) If a, b, c are distinct non-zero real numbers having the same sign. prove that

$$\text{Cot}^{-1}\left(\frac{ab+1}{a-b}\right) + \text{Cot}^{-1}\left(\frac{bc+1}{b-c}\right) + \text{Cot}^{-1}\left(\frac{ca+1}{c-a}\right) = \pi \text{ or } 2\pi.$$

(iv) If $\text{Sin}^{-1} x + \text{Sin}^{-1} y + \text{Sin}^{-1} z = \pi$, then prove that

$$x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz.$$

(v) (a) If $\text{Tan}^{-1} x + \text{Tan}^{-1} y + \text{Tan}^{-1} z = \pi$, then prove that $x + y + z = xyz$.

(b) If $\text{Tan}^{-1} x + \text{Tan}^{-1} y + \text{Tan}^{-1} z = \frac{\pi}{2}$, then prove that $xy + yz + zx = 1$.

4. Solve the following equations for x :

(i) $\text{Tan}^{-1}\left(\frac{x-1}{x-2}\right) + \text{Tan}^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

(ii) $\text{Tan}^{-1}\left(\frac{1}{2x+1}\right) + \text{Tan}^{-1}\left(\frac{1}{4x+1}\right) = \text{Tan}^{-1}\left(\frac{2}{x^2}\right)$

(iii) $3\text{Sin}^{-1}\left(\frac{2x}{1+x^2}\right) - 4\text{Cos}^{-1}\left(\frac{1-x^2}{1+x^2}\right) + 2\text{Tan}^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3}$

(iv) $\text{Sin}^{-1}(1-x) - 2\text{Sin}^{-1}x = \frac{\pi}{2}$

5. Solve the following equations.

(i) $\text{Cot}^{-1}\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\,\text{Cot}^{-1}\left(\frac{1}{x}\right)$, $x > 0$ and $x \neq 1$

(ii) $\text{Tan}\left(\text{Cos}^{-1}\frac{1}{x}\right) = \text{Sin}\left(\text{Cot}^{-1}\frac{1}{2}\right)$; $x \neq 0$

(iii) $\text{Cos}^{-1}x + \text{Sin}^{-1}\frac{x}{2} = \frac{\pi}{6}$

(iv) $\text{Cos}^{-1}\left(\sqrt{3}.x\right) + \text{Cos}^{-1}x = \frac{\pi}{2}$

(v) $\text{Sin}\left[\text{Sin}^{-1}\left(\frac{1}{5}\right) + \text{Cos}^{-1}x\right] = 1$

Key Concepts

- If $\sin\theta = x$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $\text{Sin}^{-1} x = \theta$.
- If $\cos\theta = x$ and $\theta \in [0, \pi]$, then $\text{Cos}^{-1} x = \theta$.
- If $\tan\theta = x$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then $\text{Tan}^{-1} x = \theta$.
- If $\cot\theta = x$ and $\theta \in (0, \pi)$, then $\text{Cot}^{-1} x = \theta$.
- If $\text{sce}\,\theta = x$ and $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, then $\text{Sce}^{-1} x = \theta$.
- If $\text{cosec}\,\theta = x$ and $\theta \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$, then $\text{Cosce}^{-1} x = \theta$.
- If $x \in [-1,1] - \{0\}$, then $\text{Sin}^{-1} x = \text{Cosec}^{-1} (\frac{1}{x})$.
- If $x \in [-1,1] - \{0\}$, then $\text{Cos}^{-1} x = \text{Sec}^{-1} (\frac{1}{x})$.
- (i) If $x > 0$, then $\text{Tan}^{-1} x = \text{Cot}^{-1} (\frac{1}{x})$ and

  (ii) If $x < 0$, then $\text{Tan}^{-1} x = \text{Cot}^{-1} (\frac{1}{x}) - \pi$.
- If $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, then $\text{Sin}^{-1}(\sin\theta) = \theta$ and if $x \in [-1,1]$, then $\sin(\text{Sin}^{-1} x) = x$.
- If $\theta \in [0, \pi]$, then $\text{Cos}^{-1}(\cos\theta) = \theta$ and if $x \in [-1,1]$, then $\cos(\text{Cos}^{-1} x) = x$.
- If $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then $\text{Tan}^{-1}(\tan\theta) = \theta$ and, for any $x \in \mathbf{R}$, $\tan(\text{Tan}^{-1} x) = x$.
- If $\theta \in (0, \pi)$, then $\text{Cot}^{-1}(\cot\theta) = \theta$ and, for any $x \in \mathbf{R}$, $\cot(\text{Cot}^{-1} x) = x$.

- If $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}\ \pi]$, then $\text{Sec}^{-1}(\sec\theta) = \theta$ and if $x \in (-\infty, -1] \cup [1, \infty)$, then $\sec(\text{Sec}^{-1} x) = x$.
- If $\theta \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$, then $\text{Cosec}^{-1}(\text{cosec}\,\theta) = \theta$ and if $x \in (-\infty, -1] \cup [1, \infty)$, then $\text{cosec}(\text{Cosec}^{-1} x) = x$.
- If $x \in [-1, 1]$, then $\text{Sin}^{-1}(-x) = -\text{Sin}^{-1}(x)$.
- If $x \in [-1, 1]$, then $\text{Cos}^{-1}(-x) = \pi - \text{Cos}^{-1}(x)$.
- For any $x \in \mathbf{R}$, $\text{Tan}^{-1}(-x) = -\text{Tan}^{-1}(x)$.
- For any $x \in \mathbf{R}$, $\text{Cot}^{-1}(-x) = \pi - \text{Cot}^{-1} x$.
- If $x \in (-\infty, -1] \cup [1, \infty)$, then $\text{Sec}^{-1}(-x) = \pi - \text{Sec}^{-1} x$.
- If $x \in (-\infty, -1] \cup [1, \infty)$, then $\text{Cosec}^{-1}(-x) = -\text{Cosec}^{-1} x$.
- If $\theta \in [0, \pi]$, then $\text{Sin}^{-1}(\cos\theta) = \frac{\pi}{2} - \theta$.
- If $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, then $\text{Cos}^{-1}(\sin\theta) = \frac{\pi}{2} - \theta$.
- If $\theta \in (0, \pi)$, then $\text{Tan}^{-1}(\cot\theta) = \frac{\pi}{2} - \theta$.
- If $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then $\text{Cot}^{-1}(\tan\theta) = \frac{\pi}{2} - \theta$.
- If $\theta \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$, then $\text{Sec}^{-1}(\text{cosec}\theta) = \frac{\pi}{2} - \theta$.
- If $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, then $\text{Cosec}^{-1}(\sec\theta) = \frac{\pi}{2} - \theta$.
- (i) If $0 \le x \le 1$, then $\text{Sin}^{-1} x = \text{Cos}^{-1}(\sqrt{1 - x^2})$.

  (ii) If $-1 \le x < 0$, then $\text{Sin}^{-1} x = -\text{Cos}^{-1}(\sqrt{1 - x^2})$.

(iii) If $-1 < x < 1$, $\text{Sin}^{-1} x = \text{Tan}^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$.

❖ (i) If $0 < x \le 1$, then $\text{Cos}^{-1} x = \text{Sin}^{-1}\left(\sqrt{1-x^2}\right) = \text{Tan}^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$.

(ii) If $-1 \le x < 0$, then

$$\text{Cos}^{-1} x = \pi - \text{Sin}^{-1}\left(\sqrt{1-x^2}\right) = \pi + \text{Tan}^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right).$$

❖ If $x > 0$, then $\text{Tan}^{-1} x = \text{Sin}^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) = \text{Cos}^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)$.

❖ (i) If $-1 \le x \le 1$, then $\text{Sin}^{-1} x + \text{Cos}^{-1} x = \dfrac{\pi}{2}$.

(ii) For any $x \in \mathbf{R}$, $\text{Tan}^{-1} x + \text{Cot}^{-1} x = \dfrac{\pi}{2}$.

(iii) If $x \in (-\infty, -1] \cup [1, \infty)$, then $\text{Sec}^{-1} x + \text{Cosec}^{-1} x = \dfrac{\pi}{2}$.

❖ (i) If $x, y \in [0, 1]$ and $x^2 + y^2 \le 1$, then

$$\text{Sin}^{-1} x + \text{Sin}^{-1} y = \text{Sin}^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right).$$

(ii) If $x, y \in [0, 1]$ and $x^2 + y^2 > 1$, then

$$\text{Sin}^{-1} x + \text{Sin}^{-1} y = \pi - \text{Sin}^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right).$$

(iii) If $x, y \in [0, 1]$, then $\text{Sin}^{-1} x + \text{Sin}^{-1} y = \text{Cos}^{-1}\left(\sqrt{1-x^2} . \sqrt{1-y^2} - xy\right)$.

❖ (i) If $x, y \in [0, 1]$, $\text{Sin}^{-1} x - \text{Sin}^{-1} y = \text{Sin}^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$.

(ii) If $0 \le y \le x \le 1$, then

$$\text{Sin}^{-1} x - \text{Sin}^{-1} y = \text{Cos}^{-1}\left(\sqrt{1-x^2}\ \sqrt{1-y^2} + xy\right).$$

❖ (i) If $x, y \in [0, 1]$, then

$$\text{Cos}^{-1}x + \text{Cos}^{-1}y = \text{Cos}^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right).$$

(ii) If $x, y \in [0, 1]$ and $x^2 + y^2 \geq 1$, then

$$\text{Cos}^{-1}x + \text{Cos}^{-1}y = \text{Sin}^{-1}\left(y\sqrt{1-x^2} + x\sqrt{1-y^2}\right).$$

❖ (i) If $0 \leq x \leq y \leq 1$, then

$$\text{Cos}^{-1}x - \text{Cos}^{-1}y = \text{Cos}^{-1}\left(xy + \sqrt{1-x^2}.\sqrt{1-y^2}\right).$$

(ii) If $x, y \in [0, 1]$, then

$$\text{Cos}^{-1}x - \text{Cos}^{-1}y = \text{Sin}^{-1}\left(y\sqrt{1-x^2} - x\sqrt{1-y^2}\right).$$

❖ (i) If $x > 0,\ y > 0$, then

$$\text{Tan}^{-1}x + Tan^{-1}y = \begin{cases} Tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & if \quad xy < 1 \\ \pi + Tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & if \quad xy > 1 \\ \dfrac{\pi}{2} & if \quad xy = 1 \end{cases}$$

(ii) If $x < 0,\ y < 0$, then

$$\text{Tan}^{-1}x + \text{Tan}^{-1}y = \begin{cases} Tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & if \quad xy > 1 \\ -\pi + Tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & if \quad xy < 1 \\ -\dfrac{\pi}{2} & if \quad xy = 1 \end{cases}$$

❖ If $x \,.\, y > 0$ then $\text{Tan}^{-1}x - \text{Tan}^{-1}y = \text{Tan}^{-1}\left(\dfrac{x-y}{1+xy}\right)$.

❖ If $x,\ y,\ z$ have the same sign and $xy + yz + zx < 1$, then

$$\text{Tan}^{-1}x + \text{Tan}^{-1}y + \text{Tan}^{-1}z = \text{Tan}^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right).$$

Historical Note

The power series of $\text{Tan}^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ for $|x| \le 1$ is generally known as *Gregory* (1667 AD) series, named after *James Gregory* of Scotland. *Madhava's* rule leads us to the series

$$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} \cdots$$ which is the same as *Gregory* series.

Madhava of Sangama-grama (1350 - 1425 A.D.) was a mathematician - astronomer of India (Kerala). He was the first to have developed infinite series, and approximations for a range of trigonometric functions. His discoveries opened the doors to the present mathematical analysis. His contributions to infinite series, calculus, trigonometry, geometry and algebra are noteworthy.

Answers

Exercise 8(a)

I. **1.** (i) $-\frac{\pi}{3}$ (ii) $\frac{\pi}{4}$ (iii) $\frac{3\pi}{4}$

(iv) $\frac{5\pi}{6}$ (v) 1 (vi) $\frac{\pi}{6}$

(vii) $\frac{3\pi}{4}$

2. (i) $\frac{4}{5}$ (ii) $\frac{63}{16}$ (iii) $\frac{24}{25}$

(iv) $\frac{2\pi}{7}$ (v) $\frac{5\pi}{6}$

3. (i) $\frac{x}{2}$ (ii) $\frac{\pi}{4} + \frac{x}{2}$ (iii) $\frac{|x|}{2}$

(iv) $\frac{\pi}{2}$ (v) $\left\{\frac{\pi}{4} + \frac{1}{2}\tan^{-1}x\right\}$

II. **2.** (i) $\frac{63}{65}$ (ii) $\frac{27}{11}$ (iii) $\frac{33}{65}$

5. (ii) $\frac{17}{6}$ (iii) $x = \frac{\sqrt{3}}{2}$

III. **4.** (i) $\pm\frac{1}{\sqrt{2}}$ (ii) $3, \frac{-2}{3}$ (iii) $\frac{1}{\sqrt{3}}$ (iv) 0

5. (i) $\frac{1}{\sqrt{3}}$ (ii) $\frac{3}{\sqrt{5}}$ (iii) 1 (iv) $\frac{1}{2}$ (v) $\frac{1}{5}$

Chapter 9

Hyperbolic Functions

"Weierstrass was the mathematical conscience par excellence and he became known as the father of modern analysis"

- Howard Eves

Introduction

If we take $x = a\cos\theta$ and $y = a\sin\theta\ (\theta \in \mathbf{R})$, then $x^2 + y^2 = a^2$. In other words, for any real value of θ, the point $(a\cos\theta,\ a\sin\theta)$ lies on the Circle $x^2 + y^2 = a^2$. For this reason, the trigonometric functions we have considered in chapters 6, 7 and 8 are also known as circular functions.

If we take $x = a\left(\dfrac{e^{\theta} + e^{-\theta}}{2}\right)$ and $y = b\left(\dfrac{e^{\theta} - e^{-\theta}}{2}\right)$, $(\theta \in \mathbf{R})$

then we get that $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. This is the equation of a **'hyperbola'** (Hyperbolas will be discussed in the second year intermediate course). This means that, points on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ are in the form

$$\left(a\left(\frac{e^{\theta} + e^{-\theta}}{2}\right),\ b\left(\frac{e^{\theta} - e^{-\theta}}{2}\right)\right),\ (\theta \in \mathbf{R}).$$

Weierstrass

(1815 - 1897)

Karl Weierstrass *was a German mathematician who is often cited as the "**father of modern analysis**". He was a great teacher. He brought rigour into mathematics. Weierstrassian rigour became synonymous with extremely careful reasoning.*

Keeping this fact in view, the **Hyperbolic functions** are introduced. The number e is defined as

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \infty = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

For any $x \in \mathbf{R}$, it is known that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

The number e is also given by $e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$. You will learn the proofs of these results in higher classes and you will also learn that e is an irrational number with $2 < e < 3$. The approximate value of e is given by

$$e \simeq 2.718281 \ldots$$

9.1 Definitions of Hyperbolic functions, graphs

We begin with the formal definitions of hyperbolic functions.

9.1.1(a) Definition

1. *The function* $f : \mathbf{R} \to \mathbf{R}$ *defined by*

$$f(x) = \frac{e^x - e^{-x}}{2}, \quad \text{for all} \quad x \in \mathbf{R}$$

is called the ***'hyperbolic sine'*** *function. It is denoted by* $\mathbf{sinh}\ x$. *Thus*

$$\mathbf{sinh}\ x = \frac{e^x - e^{-x}}{2} \quad \text{for all} \quad x \in \mathbf{R}$$

9.1.1(b) Definition

The function $f : \mathbf{R} \to \mathbf{R}$ *defined by*

$$f(x) = \frac{e^x + e^{-x}}{2}, \quad \text{for all} \quad x \in \mathbf{R}$$

is called the ***'hyperbolic cosine'*** *function. This is denoted by* $\mathbf{cosh}\ x$, *so that*

$$\mathbf{cosh}\ x = \frac{e^x + e^{-x}}{2} \quad \text{for all} \quad x \in \mathbf{R}$$

Now we define the other hyperbolic functions like in circular trigonometric functions.

3. $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, *for all* $x \in \mathbf{R}$.

4. $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$, *for all* $x \in \mathbf{R} \setminus \{0\}$

5. $\text{sech}\, x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$, *for all* $x \in \mathbf{R}$

6. $\text{cosech}\, x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$, *for all* $x \in \mathbf{R} \setminus \{0\}$

The six functions defined above are called ***"hyperbolic functions".***

9.1.2 Note

From the above definition, we observe the following

1. $\cosh 0 = \dfrac{e^0 + e^{-0}}{2} = 1$ and $\sinh 0 = \dfrac{e^0 - e^{-0}}{2} = 0$.

2. For any $x \in \mathbf{R}$, $\cosh(-x) = \dfrac{e^{-x} + e^x}{2} = \cosh x$.

Hence the function $f(x) = \cosh x \ (x \in \mathbf{R})$ is an **even function.**

3. For any $x \in \mathbf{R}$, $\sinh(-x) = \dfrac{e^{-x} - e^x}{2} = -\left(\dfrac{e^x - e^{-x}}{2}\right) = -\sinh x$.

Hence the function $f(x) = \sinh x \ (x \in \mathbf{R})$ is an **odd function.**

4. From (2) and (3) above we get that $\tanh x$, $\coth x$, $\text{cosech}\, x$ are odd functions and $\text{sech}\, x$ is an even function.

We have proved in earlier chapters the following identities regarding circular trigonometric functions.

$$\cos^2 x + \sin^2 x = 1, \quad \text{for all } x \in \mathbf{R}$$

$$\sec^2 x - \tan^2 x = 1, \quad \text{for all } x \in \mathbf{R} \setminus \left\{(2n+1)\frac{\pi}{2} \,\middle|\, n \in \mathbf{Z}\right\}$$

$$\text{and } \text{cosec}^2 x - \cot^2 x = 1, \quad \text{for all } x \in \mathbf{R} \setminus \{n\pi \mid n \in \mathbf{Z}\}.$$

Now we prove the following identities for hyperbolic functions.

9.1.3 Identities

1. $\cosh^2 x - \sinh^2 x = 1$, for all $x \in \mathbf{R}$.
2. $1 - \tanh^2 x = \text{sech}^2 x$, for all $x \in \mathbf{R}$.
3. $\coth^2 x - 1 = \text{cosech}^2 x$, for all $x \in \mathbf{R} \setminus \{0\}$.

Proof: 1. $\cosh^2 x - \sinh^2 x = \dfrac{(e^x + e^{-x})^2}{4} - \dfrac{(e^x - e^{-x})^2}{4}$

$$= \frac{1}{4}\left\{\left(e^{2x} + e^{-2x} + 2\right) - \left(e^{2x} + e^{-2x} - 2\right)\right\}$$

$$= \frac{1}{4} \cdot 4 = 1.$$

2. From (1) above, $\cosh^2 x - \sinh^2 x = 1$. On dividing both sides by $\cosh^2 x$, we get

$$1 - \tanh^2 x = \text{sech}^2 x.$$

3. Again from (1), we have $\cosh^2 x - \sinh^2 x = 1$.

Since $x \neq 0$, $\sinh^2 x \neq 0$. So, on dividing the above equation both sides by $\sinh^2 x$, we get $\coth^2 x - 1 = \text{cosech}^2 x$.

9.1.4 Graphs of hyperbolic functions

(i) The graph of $y = \sinh x$

Let $y = \sinh x$.

Then $y = \dfrac{e^x - e^{-x}}{2} = \dfrac{1}{2}\left(\dfrac{e^{2x} - 1}{e^x}\right)$.

To draw the graph of y = sinh x, the following observations are useful.

For $x > 0$, $e^{2x} > 1$ and hence $y > 0$.

For $x = 0$, $e^{2x} = 1$ and hence $y = 0$.

For $x < 0$, $e^{2x} < 1$ and hence $y < 0$.

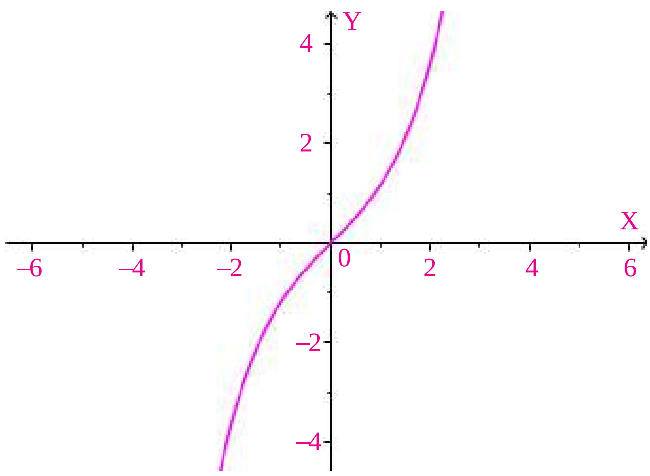


Fig. 9.1 Graph of $y = \sinh x$

Also as $x \to \infty, e^x \to \infty$ and $\frac{1}{e^x} \to 0$. So $\sinh x \to \infty$.

and as $x \to -\infty$, $e^x \to 0$ and $\frac{1}{e^x} \to \infty$. So $\sinh x \to -\infty$.

Further y is increasing with x and is continuous on **R.** Thus the graph of $y = \sinh x$ is as shown in Fig. 9.1.

(ii) The graph of $y = \cosh x$

Let $y = \cosh x$. That is $y = \frac{e^x + e^{-x}}{2} = \frac{\sqrt{\left(e^x - e^{-x}\right)^2 + 4}}{2} = \sqrt{\frac{\left(e^x - e^{-x}\right)^2}{4} + 1}$.

Thus $y \geq 1$ for all $x \in \mathbf{R}$. Also as $x \to \infty$, we get as above that $y \to \infty$. Further y is decreasing on $(-\infty, 0]$, increasing on $[0, \infty)$ and y is continuous on $\mathbf{R}$. Since the function $y = \cosh x$ is an even function, its graph is symmetric about y-axis. Keeping these points in view we can draw the graph of $y = \cosh x$ as shown in Fig. 9.2.

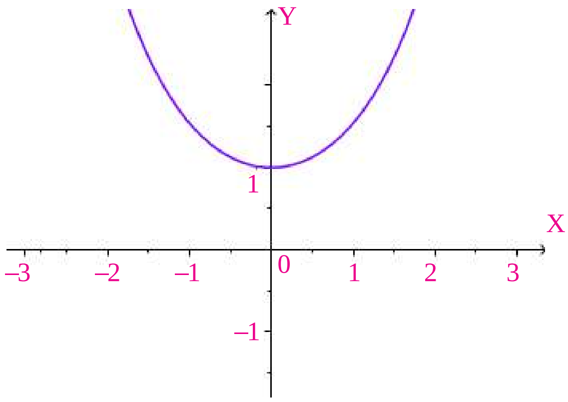


Fig. 9.2 Graph of $y = \cosh x$

(iii) The graph of $y = \tanh x$

Let $y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$. Observe the following

At $x = 0, \tanh x = 0$; For $x > 0, \tanh x > 0$ and for $x < 0, \tanh x < 0$.

As $x \to \infty$, $y = \frac{1 - \frac{1}{e^{2x}}}{1 + \frac{1}{e^{2x}}} \to 1 \left(\text{since } e^{2x} \to \infty\right)$

As $x \to -\infty$, $y = \frac{e^{2x} - 1}{e^{2x} + 1} \to -1 \left(\text{since } e^{2x} \to 0\right)$

Further, y is increasing and continuous on **R.** Now we can draw the graph of $y = \tanh x$ as shown in Fig. 9.3.

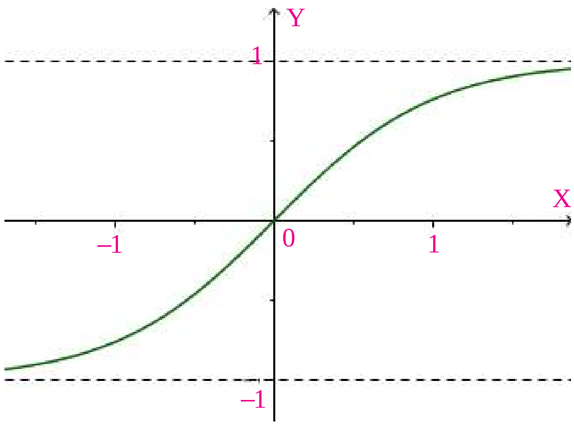


Fig. 9.3 Graph of $y = \tanh x$

(iv) The graph of $y = \coth x$

Let $y = \coth x$

$$= \frac{e^x + e^{-x}}{e^x - e^{-x}} \quad (\text{for } x \neq 0).$$

Then $y = \dfrac{e^{2x} + 1}{e^{2x} - 1}$. Observe that

as $x \to \infty$, $y \to 1$ and as $x \to -\infty$, $y \to -1$.

Also if $x > 0$, then $y > 0$ and

$y \to \infty$ as $x \to 0+$ (i.e., $x > 0$).

If $x < 0$, then $y < 0$ and

$y \to -\infty$ as $x \to 0-$ (i.e., $x < 0$).

Further, y is decreasing and continuous on **R**.

Now we draw the graph of $y = \coth x$ as shown in Fig. 9.4.

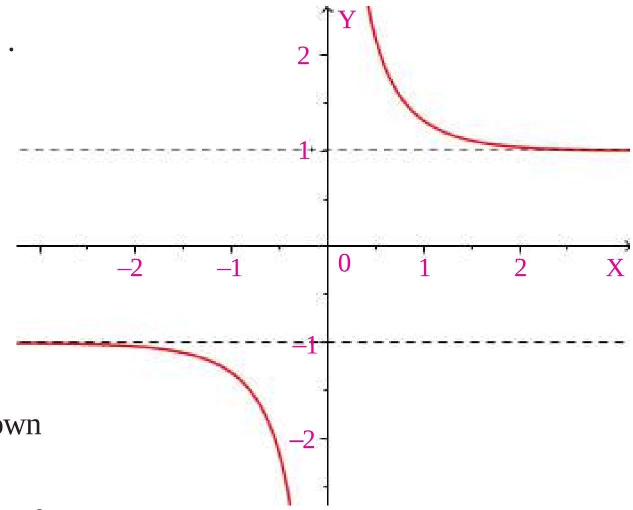


Fig. 9.4 Graph of $y = \coth x$

As above, we can also draw the graphs of $y = \text{sech}\ x$ and $y = \text{cosech}\ x$ as shown below.

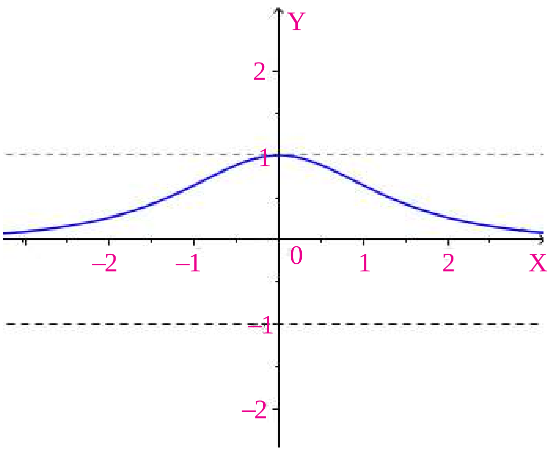


Fig. 9.5 Graph of $y = \text{sech}\ x$

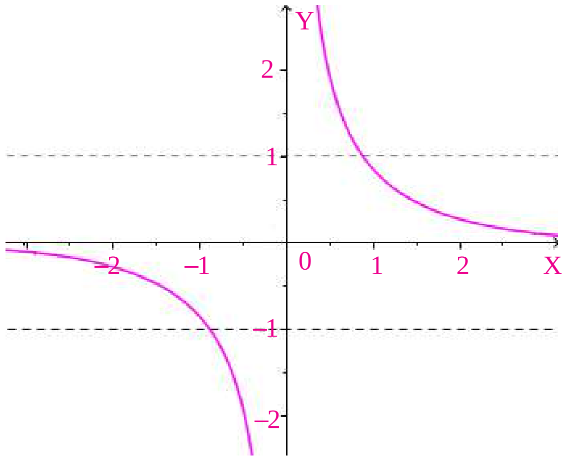


Fig. 9.6 Graph of $y = \text{cosech}\ x$

9.1.5 Domain and Range of hyperbolic functions

From the observations we have made in 9.1.4 and from the graphs, we observe that the domains and ranges of the hyperbolic functions are as given in the following table.

Table 9.1

Sl.no.	Function $y = f(x)$	Domain (x)	Range (y)
(i)	$y = \sinh x$	$\mathbf{R}$	$\mathbf{R}$
(ii)	$y = \cosh x$	$\mathbf{R}$	$[1, \infty)$
(iii)	$y = \tanh x$	$\mathbf{R}$	$(-1, 1)$
(iv)	$y = \coth x$	$\mathbf{R} \setminus \{0\}$	$(-\infty, -1) \cup (1, \infty)$
(v)	$y = \text{sech } x$	$\mathbf{R}$	$(0, 1]$
(vi)	$y = \text{cosech } x$	$\mathbf{R} \setminus \{0\}$	$\mathbf{R} \setminus \{0\}$

9.2 Definition of Inverse hyperbolic functions and graphs

In this section we define the inverses of hyperbolic functions by taking the domain suitably in such a way that the functions become bijections.

9.2.1 Definition

1. *The function* $f : \mathbf{R} \to \mathbf{R}$ *defined by* $f(x) = \sinh x$ *for all* $x \in \mathbf{R}$, *is a bijection. Thus the inverse of this function exists and it is denoted by* $\sinh^{-1}$. *Thus, if* x, y *are real numbers then*

$$\sinh^{-1} x = y \iff \sinh y = x.$$

2. *The function* $f : [0, \infty) \to [1, \infty)$ *defined by* $f(x) = \cosh x$, *for all* $x \in [0, \infty)$, *is a bijection. We define* $\cosh^{-1} : [1, \infty) \to [0, \infty)$ *by*

$$\cosh^{-1} x = y \iff \cosh y = x \quad \textit{for all } x \in [1, \infty).$$

3. *The function* $f : \mathbf{R} \to (-1, 1)$ *defined by* $f(x) = \tanh x$, *for all* $x \in \mathbf{R}$, *is a bijection. We define* $\tanh^{-1} : (-1, 1) \to \mathbf{R}$ *by*

$$\tanh^{-1} x = y \iff \tanh y = x \quad \textit{for all } x \in (-1, 1).$$

Similarly,

4. $\coth^{-1} : \mathbf{R} \setminus [-1, 1] \to \mathbf{R} \setminus \{0\}$ *is defined by*

$$\coth^{-1} x = y \iff \coth y = x \quad \textit{for all } x \in \mathbf{R} \setminus [-1, 1].$$

5. $\text{sech}^{-1} : (0, 1] \to [0, \infty)$ *is defined by*

$$\text{sech}^{-1} x = y \iff \text{sech}\, y = x \quad \textit{for all } x \in (0, 1].$$

6. $\text{cosech}^{-1} : \mathbf{R} \setminus \{0\} \to \mathbf{R} \setminus \{0\}$ *is defined by*

$$\text{cosech}^{-1} x = y \iff \text{cosech}\, y = x \quad \textit{for all } x \in \mathbf{R} \setminus \{0\}.$$

9.2.2 Domain and Range of inverse hyperbolic functions

The domains and ranges of the six inverse hyperbolic functions defined above are given in the following table.

Sl.no.	Inverse hyperbolic function $y = f(x)$	Domain (x)	Range (y)
(i)	$y = \sinh^{-1} x$	$\mathbf{R}$	$\mathbf{R}$
(ii)	$y = \cosh^{-1} x$	$[1, \infty)$	$[0, \infty)$
(iii)	$y = \tanh^{-1} x$	$(-1, 1)$	$\mathbf{R}$
(iv)	$y = \coth^{-1} x$	$\mathbf{R} \setminus [-1, 1]$	$\mathbf{R} \setminus \{0\}$
(v)	$y = \text{sech}^{-1} x$	$(0, 1]$	$[0, \infty)$
(vi)	$y = \text{cosech}^{-1} x$	$\mathbf{R} \setminus \{0\}$	$\mathbf{R} \setminus \{0\}$

9.2.3 Graphs of inverse hyperbolic functions

The graphs of the six inverse hyperbolic functions are given below.

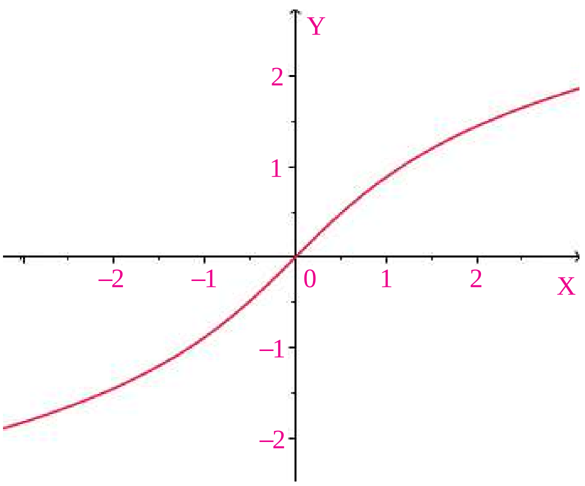


Fig. 9.7 Graph of $y = \sinh^{-1} x$

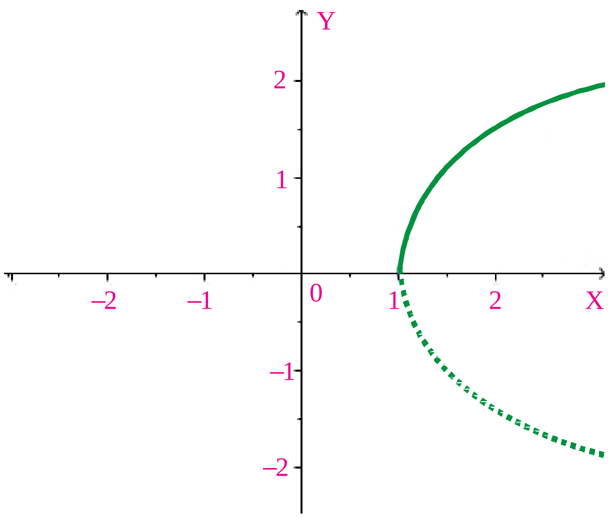


Fig. 9.8 Graph of $y = \cosh^{-1} x$

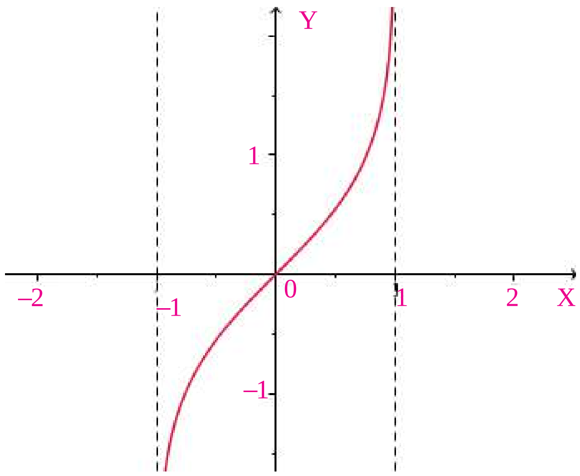


Fig. 9.9 Graph of $y = \tanh^{-1} x$

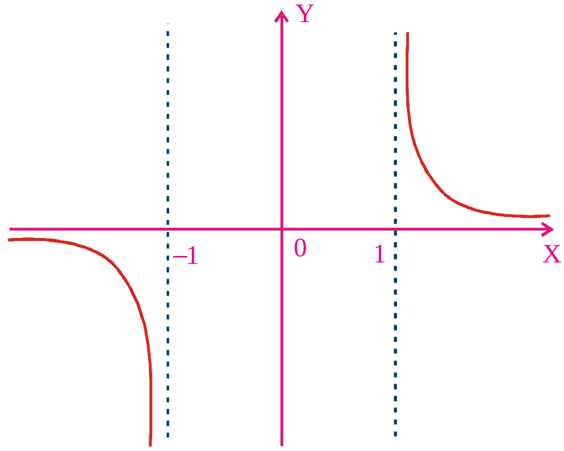


Fig. 9.10 Graph of $y = \coth^{-1} x$

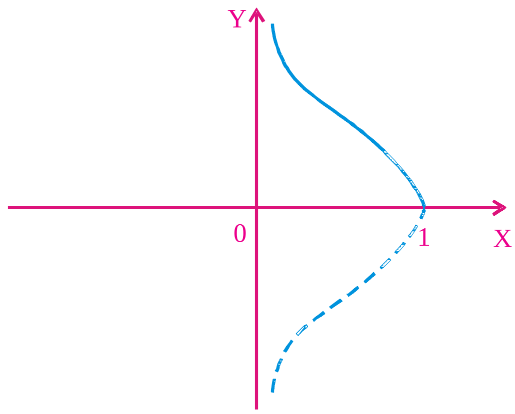


Fig. 9.11 Graph of $y = \text{sech}^{-1} x$

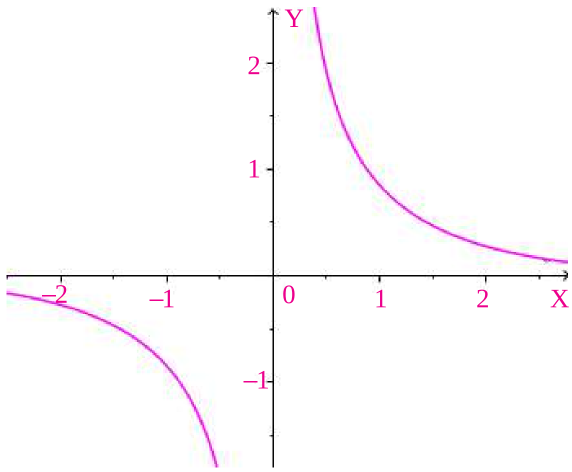


Fig. 9.12 Graph of $y = \text{cosech}^{-1} x$

9.3 Addition formulas of Hyperbolic functions

In the following we give formulae to evaluate $\sinh(x \pm y)$, $\cosh(x \pm y)$, $\tanh(x \pm y)$, $\sinh 2x$, $\cosh 2x$ and $\tanh 2x$ as in trigonometric functions.

9.3.1 Theorem: *For* $x, y \in \mathbf{R}$

(i) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

(ii) $\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$

(iii) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

(iv) $\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$

Proof : (i) $\sinh x \cosh y + \cosh x \sinh y$

$$= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)$$

$$= \frac{1}{4}\left\{e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y} + e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y}\right\}$$

$$= \frac{2\left(e^{x+y} - e^{-x-y}\right)}{4} = \frac{e^{(x+y)} - e^{-(x+y)}}{2} = \sinh\left(x+y\right).$$

$$\therefore \sinh\left(x+y\right) = \sinh x . \cosh y + \cosh x . \sinh y$$

Since $\sinh(-y) = -\sinh y$ and $\cosh(-y) = \cosh(y)$ (see note 9.1.2), on replacing y by $-y$ in (i) we get

(ii) $\sinh(x-y) = \sinh x \cosh(-y) + \cosh y \sinh(-y)$. Therefore

$$\sinh(x-y) = \sinh x . \cosh y - \cosh x . \sinh y$$

(iii) $\cosh x \cosh y + \sinh x \sinh y$

$$= \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)$$

$$= \frac{1}{4}\left\{e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y} + e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y}\right\}$$

$$= \frac{1}{4}.2.\left(e^{x+y} + e^{-x-y}\right) = \frac{e^{(x+y)} + e^{-(x+y)}}{2} = \cosh(x+y).$$ Therefore

$$\cosh(x+y) = \cosh x . \cosh y + \sinh x . \sinh y$$

On replacing y by $-y$ in (iii) above, we get

(iv) $\cosh(x-y) = \cosh x . \cosh(-y) + \sinh x . \sinh(-y)$. Therefore

$$\cosh(x-y) = \cosh x . \cosh y - \sinh x . \sinh y.$$

9.3.2 Corollary: For any $x \in \mathbf{R}$,

(i) $\sinh 2x = 2\sinh x \cosh x = \dfrac{2\tanh x}{1-\tanh^2 x}$

(ii) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x = \dfrac{1+\tanh^2 x}{1-\tanh^2 x}$

Proof

(i) On replacing y by x in 9.3.1(i), we get

$$\sinh 2x = \sinh x \,.\, \cosh x + \cosh x \sinh x = 2 \sinh x \cosh x \,.$$

Now $\sinh 2x = 2\sinh x \cosh x = \dfrac{2\sinh x \cosh x}{\cosh^2 x - \sinh^2 x}$ $(\because \cosh^2 x - \sinh^2 x = 1)$.

On dividing the numerator and denominator in R.H.S. by $\cosh^2 x$, we get

$$\boxed{\sinh 2x = \frac{2\tanh x}{1 - \tanh^2 x}}$$

(ii) On replacing y by x in 9.3.1 (iii), we get

$$\cosh 2x = \cosh x \cosh x + \sinh x \,.\, \sinh x = \cosh^2 x + \sinh^2 x \qquad \text{.... (1)}$$

Since $\cosh^2 x - \sinh^2 x = 1$, we replace $\sinh^2 x$ in (1), by $\cosh^2 x - 1$ to get

$$\boxed{\cosh 2x = 2\cosh^2 x - 1}$$

Similarly, on replacing $\cosh^2 x$ by $1 + \sinh^2 x$, we get

$$\boxed{\cosh 2x = 1 + 2\sinh^2 x}$$

Finally, $\cosh 2x = \cosh^2 x + \sinh^2 x = \dfrac{\cosh^2 x + \sinh^2 x}{\cosh^2 x - \sinh^2 x}$. $(\because \cosh^2 x - \sinh^2 x = 1)$

On dividing the numerator and denominator in R.H.S. by $\cosh^2 x$, we get

$$\boxed{\cosh 2x = \frac{1 + \tanh^2 x}{1 - \tanh^2 x}}$$

9.3.3 Theorem : *For any* $x, y \in \mathbf{R}$,

(i) $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

(ii) $\tanh(x - y) = \dfrac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$

(iii) $\coth(x + y) = \dfrac{\coth x \coth y + 1}{\coth y + \coth x}$, *if* $x \neq -y$

(iv) $\coth(x - y) = \dfrac{\coth x \coth y - 1}{\coth y - \coth x}$, *if* $x \neq y$

Proof : First we prove (i) and (iii). On replacing y by $-y$, (ii) follows from (i) and (iv) follows from (iii).

(i) $$\tanh(x+y) = \frac{\sinh(x+y)}{\cosh(x+y)} = \frac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y}$$

(by Theorem 9.3.1)

On dividing both numerator and denominator in R.H.S. by $\cosh x \ \cosh y$, we get

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

(iii) $$\coth(x+y) = \frac{\cosh(x+y)}{\sinh(x+y)} = \frac{\cosh x \cosh y + \sinh x \sinh y}{\sinh x \cosh y + \cosh x \sinh y}.$$

On dividing both numerator and denominator by $\sinh x \ \sinh y$, we get

$$\coth(x+y) = \frac{\coth x \coth y + 1}{\coth y + \coth x}$$

On replacing y by x in (i) and (iii) of theorem 9.3.3, we get the following :

9.3.4 Corollary : *For any* $x \in \mathbf{R}$,

(i) $$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$$

(ii) $$\coth 2x = \frac{\coth^2 x + 1}{2\coth x}, \text{ if } x \neq 0.$$

9.3.5 Inverse hyperbolic functions in terms of logarithmic functions

The inverse of the function $f(x) = e^x \ (x \in \mathbf{R})$ is given by $f^{-1}(x) = \log_e x \ (x > 0)$. Since hyperbolic functions are defined interms of e^x, we naturally expect formuale for inverse hyperbolic functions interms of $\log_e x$. Now we derive them in the following

9.3.6 Theorem : *For any* $x \in \mathbf{R}$,

$$\sinh^{-1} x = \log_e\left(x + \sqrt{x^2+1}\right).$$

Proof : Let $x \in \mathbf{R}$ and $y = \sinh^{-1} x$. Then

$$x = \sinh y = \frac{e^y - e^{-y}}{2} = \frac{e^{2y} - 1}{2e^y}$$

$$\Rightarrow e^{2y} - e^y(2x) - 1 = 0.$$

This is a quadratic equation in e^y. So that

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}.$$

Since $e^y > 0$ for all $y \in \mathbf{R}$ and $x < \sqrt{x^2 + 1}$, we get $e^y = x + \sqrt{x^2 + 1}$.

Thus $y = log_e\left(x + \sqrt{x^2 + 1}\right)$.

Hence $\sinh^{-1} x = \log_e\left(x + \sqrt{x^2 + 1}\right)$.

9.3.7 Theorem : For any $x \in [1, \infty)$, $\cosh^{-1} x = \log_e\left(x + \sqrt{x^2 - 1}\right)$.

Proof : Let $x \in [1, \infty)$ and $y = \cosh^{-1} x$. Then

$$x = \cosh y = \frac{e^y + e^{-y}}{2} = \frac{e^{2y} + 1}{2e^y}$$

$$\Rightarrow \left(e^y\right)^2 - 2x\left(e^y\right) + 1 = 0.$$

This is a quadratic equation in e^y. Therefore

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}.$$

But $x - \sqrt{x^2 - 1} = \dfrac{1}{x + \sqrt{x^2 - 1}} < 1$, since $x > 1$.

Further, $e^y \geq 1$ since $y \geq 0$.

Therefore, $e^y = x + \sqrt{x^2 - 1}$. That is $y = log_e\left(x + \sqrt{x^2 - 1}\right)$.

Hence, $\cosh^{-1} x = \log_e\left(x + \sqrt{x^2 - 1}\right)$.

9.3.8 Theorem : *For* $x \in (-1, 1)$, $\tanh^{-1}(x) = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$.

Proof : Let $x \in (-1, 1)$ and $y = \tanh^{-1} x$. Then

$$x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{(e^y)^2 - 1}{(e^y)^2 + 1}. \quad \dots (1)$$

Now from (1),

$$x\left((e^y)^2 + 1\right) = (e^y)^2 - 1 \Rightarrow x + 1 = (e^y)^2 (1 - x)$$

$$\Rightarrow e^{2y} = \frac{1+x}{1-x} \Rightarrow 2y = \log_e\left(\frac{1+x}{1-x}\right) \text{ (note that } \left(\frac{1+x}{1-x}\right) > 0$$

$$\text{for } x \in (-1, 1))$$

$$\Rightarrow y = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$$

Hence $\tanh^{-1} x = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$ for all $x \in (-1, 1)$.

Similarly we can prove the following.

9.3.9 Note

1. For $|x| > 1$, $\coth^{-1} x = \frac{1}{2}\log_e\left(\frac{x+1}{x-1}\right)$.

2. For $x \in (0, 1]$, $\operatorname{sech}^{-1} x = \log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right)$.

3. (i) For $x \in (-\infty, 0)$, $\operatorname{cosech}^{-1} x = \log_e\left(\frac{1-\sqrt{1+x^2}}{x}\right)$.

   (ii) For $x \in (0, \infty)$, $\operatorname{cosech}^{-1} x = \log_e\left(\frac{1+\sqrt{1+x^2}}{x}\right)$.

9.3.10 Solved Problems

1. Problem: *Prove that for any* $x \in \mathbf{R}$,

$$\sinh(3x) = 3\sinh x + 4\sinh^3 x.$$

Solution: $\sinh 3x = \sinh(2x + x)$

$$= \sinh 2x \,.\, \cosh x + \cosh 2x \sinh x$$

$$= (2\sinh x \cosh x)\cosh x + (1 + 2\sinh^2 x)\sinh x$$

$$= 2\sinh x(1 + \sinh^2 x) + (1 + 2\sinh^2 x)\sinh x \quad (\because \cosh^2 x - \sinh^2 x = 1)$$

$$= 3\sinh x + 4\sinh^3 x.$$

2. Problem: *Prove that, for any* $x \in \mathbf{R}$,

$$\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}.$$

Solution : $\tanh 3x = \tanh(2x + x) = \dfrac{\tanh 2x + \tanh x}{1 + \tanh 2x \,.\, \tanh x}$

$$= \frac{\dfrac{2\tanh x}{1 + \tanh^2 x} + \tanh x}{1 + \dfrac{2\tanh x}{1 + \tanh^2 x}.\tanh x} = \frac{2\tanh x + \tanh x(1 + \tanh^2 x)}{1 + \tanh^2 x + 2\tanh x(\tanh x)}$$

$$= \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}.$$

3. Problem: *If* $\cosh x = \dfrac{5}{2}$, *find the values of* (i) *cosh (2x)* *and* (ii) *sinh (2x)*

Solution :

(i) $\cosh(2x) = 2\cosh^2 x - 1 = 2.\dfrac{25}{4} - 1 = \dfrac{23}{2}$.

(ii) We know that $\cosh^2(2x) - \sinh^2(2x) = 1$

Therefore, $\sinh^2(2x) = \cosh^2(2x) - 1 = \left(\dfrac{23}{2}\right)^2 - 1$

$$= \frac{23^2 - 2^2}{2^2} = \frac{(25)(21)}{4}$$

$$\therefore \sinh(2x) = \pm\frac{5\sqrt{21}}{2}.$$

4. Problem: *If* $\cosh x = \sec\theta$ *then prove that* $\tanh^2 \dfrac{x}{2} = \tan^2 \dfrac{\theta}{2}$.

Solution : $\tanh^2 \dfrac{x}{2} = \dfrac{\cosh x - 1}{\cosh x + 1}$

$$= \frac{\sec\theta - 1}{\sec\theta + 1}$$

$$= \frac{1-\cos\theta}{1+\cos\theta} = \tan^2\frac{\theta}{2}.$$

5. Problem: *If* $\theta \in \left(\frac{-\pi}{4}, \frac{\pi}{4}\right)$ *and* $x = \log_e\left(\cot\left(\frac{\pi}{4}+\theta\right)\right)$, *then prove that*

(i) $\cosh x = \text{see}\, 2\theta$ *and* (ii) $\sinh x = -\tan 2\theta$.

Solution: $x = \log_e\left(\cot\frac{\pi}{4}+\theta\right) \Rightarrow e^x = \cot\left(\frac{\pi}{4}+\theta\right)$ and

$$e^{-x} = \frac{1}{\cot\left(\frac{\pi}{4}+\theta\right)} = \tan\left(\frac{\pi}{4}+\theta\right).$$

Now,

(i) $\cosh x = \dfrac{e^x + e^{-x}}{2} = \dfrac{1}{2}\left\{\cot\left(\frac{\pi}{4}+\theta\right) + \tan\left(\frac{\pi}{4}+\theta\right)\right\}$

$$= \frac{1}{2}\left\{\frac{1-\tan\theta}{1+\tan\theta} + \frac{1+\tan\theta}{1-\tan\theta}\right\} = \frac{1}{2}\left\{\frac{(1-\tan\theta)^2 + (1+\tan\theta)^2}{1-\tan^2\theta}\right\}$$

$$= \frac{1}{2}\left\{\frac{2\left(1+\tan^2\theta\right)}{1-\tan^2\theta}\right\} = \frac{1+\tan^2\theta}{1-\tan^2\theta} = \sec 2\theta.$$

(ii) $\sinh x = \dfrac{e^x - e^{-x}}{2} = \dfrac{1}{2}\left\{\cot\left(\frac{\pi}{4}+\theta\right) - \tan\left(\frac{\pi}{4}+\theta\right)\right\}$

$$= \frac{1}{2}\left\{\frac{1-\tan\theta}{1+\tan\theta} - \frac{1+\tan\theta}{1-\tan\theta}\right\} = \frac{1}{2}\left\{\frac{(1-\tan\theta)^2 - (1+\tan\theta)^2}{1-\tan^2\theta}\right\}$$

$$= \frac{1}{2}\left\{\frac{-4\tan\theta}{1-\tan^2\theta}\right\} = -\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = -\tan 2\theta.$$

6. Problem: *If* $\sinh x = 5$, *show that* $x = \log_e\left(5+\sqrt{26}\right)$.

Solution : $\sinh x = 5$

$$\Rightarrow x = \sinh^{-1}(5) = \log_e\left(5+\sqrt{5^2+1}\right) \text{ (by Theorem 9.3.6)}$$

$$= \log_e\left(5+\sqrt{26}\right).$$

7. Problem : *Show that* $\tanh^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\log_e 3$.

Solution : From Theorem 9.3.8,

$$\tanh^{-1} x = \frac{1}{2}\log_e \frac{1+x}{1-x}$$

Therefore, $\tanh^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\log_e \frac{1+\frac{1}{2}}{1-\frac{1}{2}} = \frac{1}{2}\log_e 3$.

Exercise 9(a)

1. If $\sinh x = \frac{3}{4}$, find $\cosh(2x)$ and $\sinh(2x)$.

2. If $\sinh x = 3$, then show that $x = \log_e\left(3+\sqrt{10}\right)$.

3. Prove that (i) $\tanh(x-y) = \dfrac{\tanh x - \tanh y}{1-\tanh x \tanh y}$.

   (ii) $\coth(x-y) = \dfrac{\coth x . \coth y - 1}{\coth y - \coth x}$.

4. Prove that (i) $(\cosh x - \sinh x)^n = \cosh(nx) - \sinh(nx)$, for any $n \in \mathbf{R}$.

   (ii) $(\cosh x + \sinh x)^n = \cosh(nx) + \sinh(nx)$, for any $n \in \mathbf{R}$.

5. Prove that $\dfrac{\tanh x}{\operatorname{sech} x - 1} + \dfrac{\tanh x}{\operatorname{sech} x + 1} = -2\operatorname{cosech} x$, for $x \neq 0$.

6. Prove that $\dfrac{\cosh x}{1-\tanh x} + \dfrac{\sinh x}{1-\coth x} = \sinh x + \cosh x$, for $x \neq 0$.

7. For any $x \in \mathbf{R}$, prove that $\cosh^4 x - \sinh^4 x = \cosh(2x)$.

8. If $u = \log_e\left(\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right)$ and if $\cos\theta > 0$, then prove that $\cosh u = \sec\theta$.

Key Concepts

- $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

  $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

- $\tanh x = \dfrac{\sinh x}{\cosh x}$; $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{1}{\tanh x}$,

  if $x \neq 0$; $\operatorname{sech} x = \dfrac{1}{\cosh x}$

  and $\operatorname{cosec h} x = \dfrac{1}{\sinh x}$ if $x \neq 0$.

- $\cosh^2 x - \sinh^2 x = 1$.

  $1 - \tanh^2 x = \operatorname{sech}^2 x$.

  $\coth^2 x - 1 = \operatorname{cosech}^2 x$.

- $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$.

  $\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$.

- $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$.

  $\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$.

- $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$.

  $\tanh(x - y) = \dfrac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$.

- $\sinh 2x = 2 \sinh x \cosh x = \dfrac{2\tanh x}{1 - \tanh^2 x}$.
- $\cosh 2x = \cos h^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x = \dfrac{1 + \tanh^2 x}{1 - \tanh^2 x}$.
- $\tanh 2x = \dfrac{2\tan h x}{1 + \tanh^2 x}$.
- $\sinh 3x = 3\sinh x + 4\sinh^3 x$.

  $\cosh 3x = 4\cosh^3 x - 3\cosh x$.

  $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$.
- $\sinh^{-1} x = \log_e\left(x + \sqrt{x^2 + 1}\right)$ for all $x \in \mathbf{R}$.

  $\cosh^{-1} x = \log_e\left(x + \sqrt{x^2 - 1}\right)$ for all $x \in [1, \infty)$.

  $\tanh^{-1} x = \dfrac{1}{2}\log_e\left(\dfrac{1 + x}{1 - x}\right)$ for all $x \in (-1, 1)$.

  $\coth^{-1} x = \dfrac{1}{2}\log_e\left(\dfrac{x + 1}{x - 1}\right)$ for all $x \in (-\infty, -1) \cup (1, \infty)$.

  $\text{sech}^{-1} x = \log_e\left(\dfrac{1 + \sqrt{1 - x^2}}{x}\right)$ for all $x \in (0, 1)$.

  $\text{cosceh}^{-1} x = \log_e\left(\dfrac{1 - \sqrt{1 + x^2}}{x}\right)$ if $x < 0$ and

  $= \log_e\left(\dfrac{1 + \sqrt{1 + x^2}}{x}\right)$ if $x > 0$.

Historical Note

Weierstrass (1815 - 1897), was a very influential teacher and his meticulously prepared lectures established a standard for many future mathematicians. He has devised tests for convergence of series and contributed to the theory of periodic functions, functions of real variables, elliptic functions, hyperbolic functions, convergence of infinite products and the calculus of variations. He also advanced the theory of bilinear and quadratic forms. He initiated a remarkable programme known as the "arithmetization of analysis", which stressed that all of mathematical analysis can be logically derived from the postulates of the real number system.

Answers

Exercise 9(a)

1. $\frac{17}{8}, \frac{15}{8}$

Chapter 10

Properties of Triangles

"The mathematical sciences exhibit particularly order, symmetry and limitation and these are the greatest forms of the beautiful"

– *Aristotle*

Introduction

Geometry is a branch of mathematics which investigates the relations and properties of solids, surfaces and angles. Trigonometry is based on the study of the relations between the sides and angles of a triangle. Many problems whose solutions can't be found by the methods of geometry are readily solved with the aid of trigonometry.

Hipparchus (140 B.C), a Greek mathematician established the relationship between the sides and angles of any triangle. The three most used ratios to solve a right angled triangle are the sine, the cosine and the tangent. As the angle changes in magnitude (size), the above ratios of an angle change in numerical value.

Ceva

(1647 - 1736)

***Giovanni Ceva** was an Italian mathematician widely known for proving a theorem known after him as Ceva's theorem on a property of a triangle.*

We have so far considered trigonometry as a subject useful to study the trigonometic functions and their properties in a modern view point. But one of the main aims of learning trigonometry is to determine the relation between the sides and angles of a given triangle. If the three sides are known, then the three angles can be determined and the triangle is fixed.

However, if the three angles are known, the sides cannot be fixed and the triangle is not determined. The purpose of this chapter is to develop the necessary rules and methods for determining the rest of the sides and angles of a triangle, given one or two sides and / or angles.

10.1 Relation between the sides and angles of a triangle

In triangle ABC, we denote the sides BC, CA and AB (as well as their magnitudes) by the symbols *a, b, c* respectively and the angles at the vertices i.e., $\angle CAB$, $\angle ABC$, $\angle BCA$ by the symbol A, B, C respectively. We also denote its area by the symbol Δ and its perimeter with $2s$, which is equal to $a + b + c$.

We know from elementary geometry that in any two triangles, if the corresponding angles are equal, they are similar. Similarly in two right angled triangles, if one of the acute angles in a triangle is equal to an acute angle of the other, then the two triangles are similar. From this, once the angles of a triangle are known, by just knowing one side, it is possible to determine the triangle by computing the rest of the sides in terms of trigonometric functions. Further, if any two sides of a right angled triangle are known, it is possible to determine the third side using the Pythagoras theorem and thereby fix the triangle.

In general, to construct a triangle, we need either two angles and a side or all the three sides. If two sides and the included angle are given (for example a, b, θ) the third side can be determined using the cosine rule $c^2 = a^2 + b^2 - 2ab\cos\theta$. If all the three sides of a triangle are known, then the cosine rules can still be used to fix the angles of the triangle. If θ is a right angle, this rule coincides with the Pythogoras theorem. If one side and two angles are given, the sine rule (which is discussed in the following section) can be used to solve the triangle. In using the cosine rule, one has to find the square root and this difficulty can be overcome by using the appropriate tangent law.

10.2 Sine, Cosine and Tangent Rules - Projection Rules

The circle passing through the three vertices A, B, C of ΔABC is called the circumcircle. The centre and radius of this circle are called the circumcentre and circum radius respectively. We know that the perpendicular bisectors of the sides of a triangle are concurrent and the point of their concurrence is the circumcentre. We denote the circum centre by S and circumradius by R.

The equation form of the law of sines is actually three equations, each of which is based on the proportionality of two sides of a triangle to the sines of the angles opposite to them. A study of these equations shows that the following cases of triangles can be solved by means of law of sines: (i) given any two angles and any side, (ii) given two sides and angle opposite to one of them.

The law of cosines provides relations which solve triangles coming under the cases: (iii) given two sides and the included angle, (iv) given 3 sides. In case (iv) it is possible to find anyone of the 3 angles using

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

In a triangle ABC, as usual, we denote (the magnitudes of) the sides AB, BC, CA by c, a, b respectively and the angles $\angle BAC, \angle CBA, \angle ACB$ by simply A, B, C or $\angle A, \angle B, \angle C$ respectively.

10.2.1 Theorem: *In* ΔABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$,

where R is the circumradius.

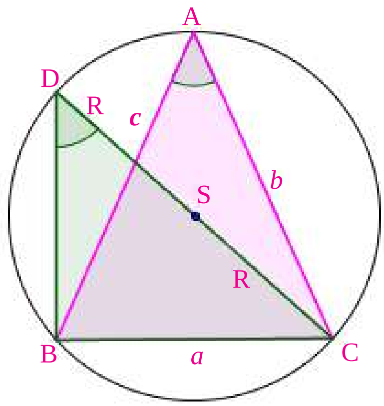


Fig. 10.1

Case (i) : $\angle A$ is acute (see Fig. 10.1).

S is the centre of the circumcircle and

CD is its diameter.

Then CS = SD = R and CD = 2R. Join BD.

Then $\angle DBC = \dfrac{\pi}{2}$ and ΔDBC is a right angled triangle.

Then $\angle BAC = \angle BDC$, ($\because$ angles in the same segment)

$$\therefore \sin A = \sin \angle BAC = \sin \angle BDC = \frac{BC}{CD} = \frac{a}{2R}.$$

$$\therefore \frac{a}{\sin A} = 2R.$$

Case (ii) : $\angle A$ is a right angle (see Fig. 10.2).

Then BC = a = 2R = 2R .1 = 2R sin 90^0

$\therefore a = 2R \sin A$. Hence $\dfrac{a}{\sin A} = 2R$.

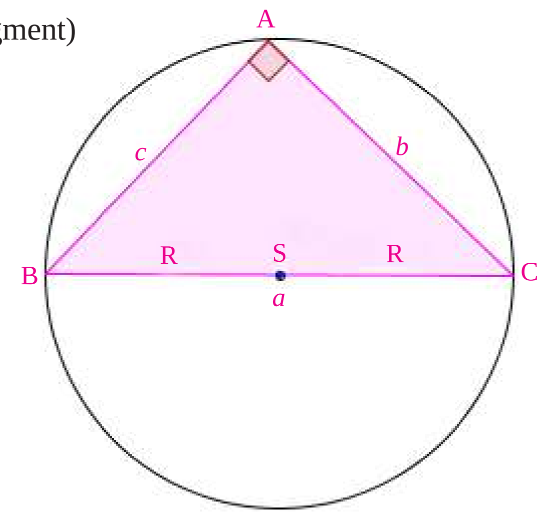


Fig. 10.2

Case (iii) : $\underline{|A}$ is obtuse (see Fig. 10.3).

$\underline{|DBC}$ is right angle. ($\because$ angle in the semi circle)

In the cyclic quadrilateral BACD,

$$\underline{|BDC} = 180^0 - \underline{|BAC} = 180^0 - A$$

In ΔBDC, $\sin A = \sin\left(180^0 - A\right)$

$$= \sin \underline{|BDC} = \frac{BC}{CD} = \frac{a}{2R}.$$

Hence $\dfrac{a}{\sin A} = 2R$.

In a similar way, we can prove

$$\frac{b}{\sin B} = 2R, \quad \frac{c}{\sin C} = 2R$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

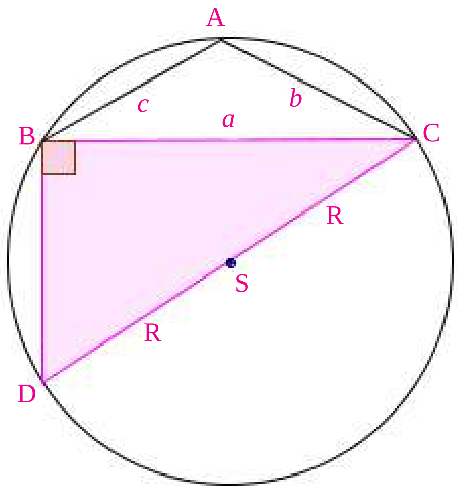


Fig. 10.3

10.2.2 Note

(i) Theorem (10.2.1) is called the **'sine rule'** or **'law of sines'**. Also in a right angled triangle, Hypotenuse = 2 (circum radius) = circum diameter.

(ii) $a = 2R \sin A$, $b = 2R \sin B$, $c = 2R \sin C$.

(or) $\sin A = \dfrac{a}{2R}$, $\sin B = \dfrac{b}{2R}$, $\sin C = \dfrac{c}{2R}$.

We shall now derive the cosine rule connecting the sides a, b, c of ΔABC with the cosines of its angles A, B, C.

10.2.3 Theorem : In ΔABC, $b^2 = c^2 + a^2 - 2\,ca \cos B$

$$c^2 = a^2 + b^2 - 2\,ab \cos C$$
$$a^2 = b^2 + c^2 - 2\,bc \cos A$$

Proof: $a^2 = (2R \sin A)^2$

$= 4R^2 [\sin (B + C)]^2$

$= 4R^2(\sin B \cos C + \cos B \sin C)^2$

$= 4R^2\{\sin^2 B(1 - \sin^2 C) + \sin^2 C (1 - \sin^2 B) + 2\sin B \sin C \cos B \cos C\}$

$= 4R^2\{\sin^2 B + \sin^2 C + 2 \sin B \sin C (\cos B \cos C - \sin B \sin C)\}$

$= 4R^2\{\sin^2 B + \sin^2 C + 2\sin B \sin C \cos (B + C)\}$

$= b^2 + c^2 - 2bc \cos A.$

The proofs of the other two results are similar.

Alternative method

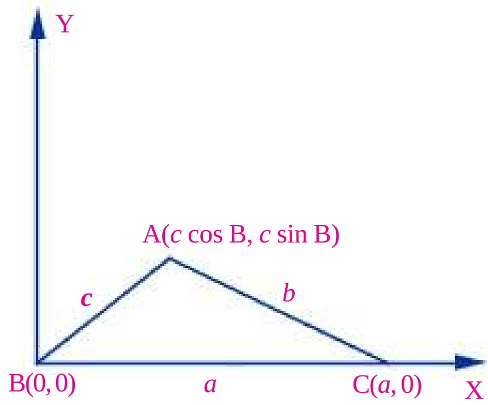


Fig. 10.4

Take the vertex B of Δ ABC as origin and its side BC along X-axis as shown in Fig. 10.4.

Then B = (0, 0) and C = (a, 0).

Angle made by the side AB with X-axis = B

Here AB = c and A = (c cos B, c sin B)

$$b^2 = CA^2 = (c\cos B - a)^2 + (c\sin B - 0)^2$$

$$= c^2\cos^2 B + a^2 - 2ca\cos B + c^2\sin^2 B$$

$$= c^2(\cos^2 B + \sin^2 B) + a^2 - 2ca\cos B$$

$$\therefore b^2 = c^2 + a^2 - 2ca\cos B.$$

Similarly we can prove that $c^2 = a^2 + b^2 - 2ab\cos C$ and $a^2 = b^2 + c^2 - 2bc\cos A$.

10.2.4 Note

(i) Theorem (10.2.3) is known as the '**law of cosines**' and the rules in it are called '**cosine rules**'.

(ii) From the cosine rules, we can write $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$

and $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

These rules are used to find the three angles of a triangle when its sides are given.

10.2.5 Theorem: *In* ΔABC, $a = b\cos C + c\cos B$.

Proof: From the cosine rules, we have

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca},\ \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\therefore b\cos C + c\cos B = b\left(\frac{a^2 + b^2 - c^2}{2ab}\right) + c\left(\frac{c^2 + a^2 - b^2}{2ca}\right)$$

$$= \frac{a^2 + b^2 - c^2 + c^2 + a^2 - b^2}{2a} = \frac{2a^2}{2a} = a.$$

Similarly, we can prove that $b = c\cos A + a\cos C$ and $c = a\cos B + b\cos A$

Note: These three rules are called the '**projection rules**'.

10.2.6 Theorem: *In* ΔABC, $\tan\left(\dfrac{B - C}{2}\right) = \dfrac{b - c}{b + c}\cot\dfrac{A}{2}$.

Proof : From the sine rule, b = 2R sin B, c = 2R sin C.

$$\text{Hence } \frac{b-c}{b+c} = \frac{2R(\sin B - \sin C)}{2R(\sin B + \sin C)} = \frac{\sin B - \sin C}{\sin B + \sin C}$$

$$= \frac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}$$

$$= \cot\left(\frac{B+C}{2}\right)\tan\left(\frac{B-C}{2}\right)$$

$$= \tan\frac{A}{2}\tan\left(\frac{B-C}{2}\right) \qquad \left[\because \cot\left(\frac{B+C}{2}\right) = \tan\frac{A}{2}\right]$$

$$\therefore \tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}.$$

In a similar way, we can prove that $\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\frac{B}{2}$

and $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$.

Note: These three results are called **'Napier analogy'** or **'tangent rules'**.

10.3 Half angle formulae and area of a triangle

We have learnt in elementary geometry that, if the base b and the altitude h are given, the area of the triangle, denoted by Δ is equal to $\frac{1}{2}bh$. However, if the three sides a, b and c are given, then $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$. In this section Δ is obtained in terms of two of its sides and the sine of the angle between them. We also obtain some relations involving half angles, the perimeter and the sides of the triangle.

10.3.1 Theorem : *In* ΔABC

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$ and

(iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$.

Proof: (i) We have $1 - \cos A = 2\sin^2\frac{A}{2}$ ... (1)

By cosine rule, $1 - \cos A = 1 - \frac{b^2+c^2-a^2}{2bc}$

$$= \frac{2bc - b^2 - c^2 + a^2}{2bc} = \frac{a^2 - (b-c)^2}{2bc}$$

$$= \frac{(a+b-c)(a-b+c)}{2bc}.$$

Since $a+b+c = 2s,\ a+b-c = 2s - 2c = 2(s-c)$

Similarly $a - b + c = 2s - 2b = 2(s-b)$

$$\therefore\ 1 - \cos A = \frac{2(s-c) \,.\, 2(s-b)}{2bc} = \frac{2(s-b)(s-c)}{bc}$$

$$\therefore\ \text{From (1)},\ 2\sin^2\frac{A}{2} = \frac{2(s-b)\,.\,(s-c)}{bc}$$

$$\therefore\ \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}. \qquad \left(\because \frac{A}{2} < 90^0,\ \sin\frac{A}{2} > 0\right)$$

(ii) We have $1 + \cos A = 2\cos^2\frac{A}{2}$ ... (2)

By cosine rule $1 + \cos A = 1 + \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{(b+c)^2 - a^2}{2bc}$

$$= \frac{(b+c+a)(b+c-a)}{2bc} = \frac{2s\,.\,2(s-a)}{2bc} = \frac{2s(s-a)}{bc}$$

From (2), $2\cos^2\dfrac{A}{2} = \dfrac{2s(s-a)}{bc}$ $\therefore \cos^2\dfrac{A}{2} = \dfrac{s(s-a)}{bc} \Rightarrow \cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$.

(iii) $\tan\dfrac{A}{2} = \dfrac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \sqrt{\dfrac{(s-b)(s-c)}{bc}} \Big/ \sqrt{\dfrac{s(s-a)}{bc}}$, (from results (i) and (ii))

$$= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

10.3.2 Note: In a similar way, we can prove that

$$\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}, \qquad \sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}, \qquad \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

$$\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}\ ,\quad \tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}\ .$$

10.3.3 Deductions: The following deductions can be made from Theorem (10.3.1):

(i) $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}$

$$=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}\,.$$

(ii) $\sin B = \frac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)}\,.$

(iii) $\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}\,.$

We now find the area of the Δ ABC denoted by the symbol Δ, in terms of two of its sides and the sine of the anlge between them.

10.3.4 Theorem : Δ = area of ΔABC

$$=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\frac{1}{2}ab\sin C$$

Proof : In ΔABC, from A draw AD perpendicular to BC (Fig.10.5).

Then $\Delta = \frac{1}{2}(\text{base})(\text{height})$

$$=\frac{1}{2}BC\,.\,AD=\frac{1}{2}a\,.AD$$

In Δ ABD, $\sin B = \frac{AD}{AB}$.

Hence AD = AB sin B = $c\sin B$

$$\Rightarrow \Delta=\frac{1}{2}a\,.(c\sin B)=\frac{1}{2}ca\sin B.$$

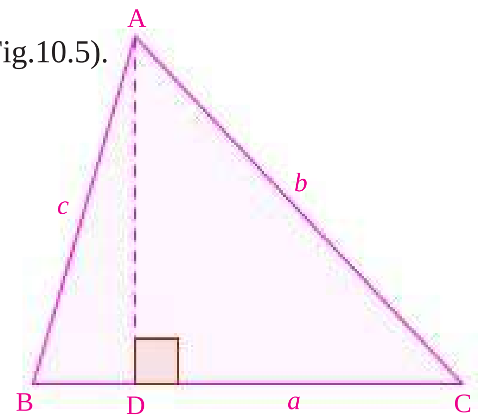


Fig. 10.5

Similarly, we can prove that $\Delta = \frac{1}{2}ab\sin C$ and $\Delta = \frac{1}{2}bc\sin A$.

$$\Rightarrow \Delta=\frac{1}{2}ab\sin C=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B \qquad \text{... (1)}$$

Consequences: $\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}(2R\sin A)(2R\sin B)\sin C$

$$=2R^2\sin A\sin B\sin C \qquad \text{... (2)}$$

$$\Delta=\frac{1}{2}ab\sin C=\frac{1}{2}ab\,.\,\frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$$

$$=\sqrt{s(s-a)(s-b)(s-c)} \qquad \text{... (3)}$$

$$\Delta = \frac{1}{2}ab \sin C = \frac{1}{2}ab \cdot \frac{c}{2R}, \left(\because \frac{c}{\sin C} = 2R\right)$$

$$= \frac{abc}{4R}. \quad \ldots (4)$$

Formulae (1) - (4) are useful for finding the area of a triangle.

10.3.5 Note

(i) $\tan\frac{A}{2}$ can also be expressed as

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \times 1 = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \cdot \sqrt{\frac{(s-b)(s-c)}{(s-b)(s-c)}}$$

$$= \frac{(s-b)(s-c)}{\sqrt{s(s-a)(s-b)(s-c)}} = \frac{(s-b)(s-c)}{\Delta}$$

$$\Rightarrow \cot\frac{A}{2} = \frac{\Delta}{(s-b)(s-c)}.$$

Similarly, we can deduce that

$$\tan\frac{B}{2} = \frac{(s-c)(s-a)}{\Delta}, \text{ hence } \cot\frac{B}{2} = \frac{\Delta}{(s-c)(s-a)}.$$

$$\text{and } \tan\frac{C}{2} = \frac{(s-a)(s-b)}{\Delta}, \text{ hence } \cot\frac{C}{2} = \frac{\Delta}{(s-a)(s-b)}.$$

(ii) $\tan\frac{A}{2}$ can also be expressed in an alternate form

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \times 1 = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \times \sqrt{\frac{s(s-a)}{s(s-a)}}$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s(s-a)} = \frac{\Delta}{s(s-a)}$$

$$\Rightarrow \cot\frac{A}{2} = \frac{s(s-a)}{\Delta}.$$

Similarly, we can deduce that

$$\tan\frac{B}{2} = \frac{\Delta}{s(s-b)} \Rightarrow \cot\frac{B}{2} = \frac{s(s-b)}{\Delta},$$

$$\tan\frac{C}{2} = \frac{\Delta}{s(s-c)} \Rightarrow \cot\frac{C}{2} = \frac{s(s-c)}{\Delta}.$$

10.3.6 Theorem : *In* Δ ABC, $\dfrac{a+b}{c} = \dfrac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$.

Proof: From sine rule, we have $a = 2R\sin A,\ b = 2R\sin B,\ c = 2R\sin C$.

Hence $\dfrac{a+b}{c} = \dfrac{2R(\sin A + \sin B)}{2R\sin C} = \dfrac{\sin A + \sin B}{\sin C}$

$$= \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\sin\frac{C}{2}\cos\frac{C}{2}} = \frac{\sin\left(90^0 - \frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}\cos\frac{C}{2}},$$

$$\left(\because A + B + C = 180^0,\ \frac{A+B}{2} = \frac{180^0 - C}{2} = 90^0 - \frac{C}{2}\right)$$

$$= \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$$

Similarly, we can prove that $\dfrac{b+c}{a} = \dfrac{\cos\left(\frac{B-C}{2}\right)}{\sin\frac{A}{2}};\ \dfrac{c+a}{b} = \dfrac{\cos\left(\frac{C-A}{2}\right)}{\sin\frac{B}{2}}$.

Note:

$$\frac{a-b}{c} = \frac{\sin\left(\frac{A-B}{2}\right)}{\cos\frac{C}{2}},\ \frac{b-c}{b} = \frac{\sin\left(\frac{B-C}{2}\right)}{\cos\frac{A}{2}} \text{ and } \frac{c-a}{b} = \frac{\sin\left(\frac{C-A}{2}\right)}{\cos\frac{B}{2}}.$$

10.3.7 Solved Problems

1. Problem: *In* ΔABC, *if* $a = 3, b = 4$ *and* $\sin A = \frac{3}{4}$, *find angle* B.

Solution: From sine rule, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$.

$$\therefore \sin B = \frac{b\sin A}{a} = \frac{4.\left(\frac{3}{4}\right)}{3} = 1 \Rightarrow B = 90^0.$$

2. Problem: *If the lengths of the sides of a triangle are 3, 4, 5, find the circumradius of the triangle.*

Solution: Given that the sides of a triangle are 3, 4, 5.

Now $3^2 + 4^2 = 5^2$. Hence the triangle is right angled and its hypotenuse = 5.

$$\therefore \text{ Circumradius } = \frac{1}{2} \text{ (hypotenuse) } = \frac{5}{2}$$

(OR)

By using Sine Rule $\frac{c}{\sin C} = 2R$

$\therefore \quad c = 2R \text{ Sin C}$

$= 2R \sin 90^0$

$c = 2R$

$R = \frac{c}{2} = \frac{5}{2}.$

3. Problem: *If* $a = 6, \ b = 5, \ c = 9,$ *then find angle* A.

Solution: From cosine rule, $\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{25+81-36}{2.5.9} = \frac{70}{90} = \frac{7}{9}.$

$$\therefore A = \text{Cos}^{-1}\left(\frac{7}{9}\right).$$

4. Problem: *In* ΔABC, *show that* $\Sigma(b+c)\cos A = 2s$.

Solution: L.H.S. $= (b + c) \cos A + (c + a) \cos B + (a + b) \cos C$

$= (b \cos A + a \cos B) + (c \cos B + b \cos C) + (a \cos C + c \cos A)$

$= c + a + b = 2s. = \text{R.H.S}.$

5. Problem: *If the sides of a triangle are* 13, 14, 15, *then find the circum diameter.*

Solution: Let $a = 13, b = 14, c = 15$. Then $2s = a + b + c = 13 + 14 + 15 = 42,$

$s = 21, s - a = 8, s - b = 7, s - c = 6.$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21.8.7.6} = 84.$$

$$\Rightarrow \frac{abc}{4R} = \Delta = 84 \Rightarrow 4R \times 84 = 13 \times 14 \times 15.$$

$$\therefore \text{ Circum diameter (2R) } = \frac{65}{4}.$$

6. Problem: *In* ΔABC, *if* $(a + b + c)(b + c - a) = 3bc$, *find* A.

Solution: $2s(2s - 2a) = 3bc \Rightarrow \frac{s(s-a)}{bc} = \frac{3}{4}$

$$\Rightarrow \cos^2 \frac{A}{2} = \frac{3}{4} \Rightarrow \cos \frac{A}{2} = \frac{\sqrt{3}}{2} = \cos 30^0.$$

$$\therefore \frac{A}{2} = 30^0 \Rightarrow A = 60^0.$$

7. Problem: *If* $a = 4, b = 5, c = 7$, *find* $\cos \frac{B}{2}$.

Solution: $2s = a + b + c = 4 + 5 + 7 = 16 \Rightarrow s = 8, s - b = 3$

$$\therefore \cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}} = \sqrt{\frac{8(3)}{7(4)}} = \sqrt{\frac{6}{7}}.$$

8. Problem: *In* ΔABC, *find* $b\cos^2 \frac{C}{2} + c \cos^2 \frac{B}{2}$.

Solution: $b \cos^2 \frac{C}{2} + c \cos^2 \frac{B}{2} = b\left[\frac{s(s-c)}{ab}\right] + c\left[\frac{s(s-b)}{ca}\right]$

$$= \frac{s(s-c)}{a} + \frac{s(s-b)}{a} = \frac{s}{a}[s - c + s - b] = \frac{s}{a} . a = s.$$

9. Problem: *If* $\tan \frac{A}{2} = \frac{5}{6}$ *and* $\tan \frac{C}{2} = \frac{2}{5}$, *determine the relation between a, b, c.*

Solution: $\tan \frac{A}{2} \cdot \tan \frac{C}{2} = \frac{5}{6} \cdot \frac{2}{5} = \frac{2}{6}$.

$$\text{i.e., } \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \sqrt{\frac{(s-b)(s-a)}{s(s-c)}} = \frac{2}{6}.$$

$$\Rightarrow \frac{s-b}{s} = \frac{1}{3} \Rightarrow 3s - 3b = s \Rightarrow 2s = 3b.$$

$\Rightarrow a + b + c = 3b \Rightarrow a + c = 2b$. Hence a, b, c are in A.P.

10. Problem: *If* $\cot \frac{A}{2} = \frac{b+c}{a}$, *find angle* B.

Solution: $\cot \frac{A}{2} = \frac{b+c}{a} \Rightarrow \frac{\cos \frac{A}{2}}{\sin \frac{A}{2}} = \frac{\cos\left(\frac{B-C}{2}\right)}{\sin \frac{A}{2}}$, (by 10.3.6)

$$\Rightarrow \frac{A}{2} = \frac{B-C}{2} \Rightarrow A = B - C \Rightarrow A + C = B \Rightarrow A + B + C = 2B$$

$\therefore 2B = 180^0 \Rightarrow B = 90^0$.

11. Problem: *If* $\tan\left(\frac{C - A}{2}\right) = k \cot \frac{B}{2}$, *find* k.

Solution: Comparing with $\tan\left(\frac{C-A}{2}\right) = \left(\frac{c-a}{c+a}\right)\cot \frac{B}{2}$ (by tangent law),

we get that $k = \frac{c-a}{c+a}$.

12. Problem: *In* ΔABC, *show that* $\dfrac{b^2 - c^2}{a^2} = \dfrac{\sin(B - C)}{\sin(B + C)}$.

Solution: L.H.S. = $\dfrac{b^2 - c^2}{a^2} = \dfrac{4R^2(\sin^2 B - \sin^2 C)}{4R^2 \sin^2 A}$

$$= \frac{\sin^2 B - \sin^2 C}{\sin^2 A} = \frac{\sin(B + C)\sin(B - C)}{\sin^2(B + C)}, \quad [\because \sin A = \sin(B + C)]$$

$$= \frac{\sin(B - C)}{\sin(B + C)} = \text{R.H.S.}$$

13. Problem: *Show that* $a^2 \cot A + b^2 \cot B + c^2 \cot C = \dfrac{abc}{R}$.

Solution: L.H.S. = $a^2 \cot A + b^2 \cot B + c^2 \cot C$

$$= 4R^2 \sin^2 A \cdot \frac{\cos A}{\sin A} + 4R^2 \sin^2 B \cdot \frac{\cos B}{\sin B} + 4R^2 \sin^2 C \cdot \frac{\cos C}{\sin C},$$

(by sine rule)

$$= 2R^2(2 \sin A \cos A + 2 \sin B \cos B + 2 \sin C \cos C)$$

$$= 2R^2(\sin 2A + \sin 2B + \sin 2C)$$

$$= 2R^2(4 \sin A \sin B \sin C)$$

$$= \frac{1}{R}(2R \sin A)(2R \sin B)(2R \sin C) = \frac{abc}{R} = \text{R.H.S.}$$

14. Problem: *Show that* $(b - c)^2 \cos^2 \dfrac{A}{2} + (b + c)^2 \sin^2 \dfrac{A}{2} = a^2$.

Solution: L.H.S. = $(b^2 + c^2 - 2bc)\cos^2 \dfrac{A}{2} + (b^2 + c^2 + 2bc)\sin^2 \dfrac{A}{2}$

$$= (b^2 + c^2)\left(\cos^2 \frac{A}{2} + \sin^2 \frac{A}{2}\right) - 2bc\left(\cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}\right)$$

$$= b^2 + c^2 - 2bc \cos A = a^2.$$

15. Problem: *Prove that* $a(b \cos C - c \cos B) = b^2 - c^2$.

Solution: L.H.S. = $ab \cos C - ca \cos B$

$$= \left(\frac{a^2 + b^2 - c^2}{2}\right) - \left(\frac{c^2 + a^2 - b^2}{2}\right), \text{ (by cosine rule)}$$

$$= \frac{1}{2}\left[a^2 + b^2 - c^2 - c^2 - a^2 + b^2\right] = b^2 - c^2 = \text{R.H.S.}$$

16. Problem: *Show that* $\dfrac{c - b\cos A}{b - c\cos A} = \dfrac{\cos B}{\cos C}$.

Solution: From projection rule, $c = a\cos B + b\cos A$ and $b = c\cos A + a\cos C$.

$$\text{Now} \quad \text{L.H.S.} = \frac{(a\cos B + b\cos A) - b\cos A}{(c\cos A + a\cos C) - c\cos A}$$

$$= \frac{a\cos B}{a\cos C} = \frac{\cos B}{\cos C} = \text{R.H.S.}$$

17. Problem: *In* ΔABC, *if* $\dfrac{1}{a+c} + \dfrac{1}{b+c} = \dfrac{3}{a+b+c}$, *show that* $C = 60^0$.

Solution: $\dfrac{1}{a+c} + \dfrac{1}{b+c} = \dfrac{3}{a+b+c} \Rightarrow \dfrac{b+c+a+c}{(a+c)(b+c)} = \dfrac{3}{a+b+c}$

$$\Rightarrow 3(a+c)(b+c) = (a+b+2c)(a+b+c)$$

$$\Rightarrow 3(ab + ac + bc + c^2) = (a^2 + b^2 + 2ab) + 3c(a+b) + 2c^2$$

$$\Rightarrow ab = a^2 + b^2 - c^2 = 2ab\cos C \text{ (from cosine rule)}$$

$$\Rightarrow \cos C = \frac{1}{2} \Rightarrow C = 60^0.$$

18. Problem: *If* $a = (b-c)\sec\theta$, *prove that* $\tan\theta = \dfrac{2\sqrt{bc}}{b-c}\sin\dfrac{A}{2}$.

Solution: $a = (b-c)\sec\theta \Rightarrow \sec\theta = \dfrac{a}{b-c}$

$$\tan^2\theta = \sec^2\theta - 1 = \frac{a^2}{(b-c)^2} - 1 = \frac{a^2 - (b-c)^2}{(b-c)^2}$$

$$= \frac{(a+b-c)(a-b+c)}{(b-c)^2}$$

$$= \frac{2(s-c)\,2(s-b)}{(b-c)^2} = \frac{4bc}{(b-c)^2}\frac{(s-b)(s-c)}{bc}$$

$$= \frac{4bc}{(b-c)^2} \cdot \sin^2\frac{A}{2}$$

$$\therefore \tan\theta = \frac{2\sqrt{bc}}{b-c}\sin\frac{A}{2}.$$

19. Problem: *In* ΔABC, *show that* $(a+b+c)\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=2c\cot\frac{C}{2}$.

Solution: L.H.S. $= (a+b+c)\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right) = 2s\left[\frac{\Delta}{s(s-a)}+\frac{\Delta}{s(s-b)}\right]$

$= 2\Delta\left[\frac{1}{s-a}+\frac{1}{s-b}\right] = 2\Delta\left[\frac{s-b+s-a}{(s-a)(s-b)}\right]$

$= 2c\left[\frac{\Delta}{(s-a)(s-b)}\right] = 2c\cot\frac{C}{2} =$ R.H.S.

20. Problem: *Show that* $b^2\sin 2C + c^2\sin 2B = 2bc\sin A$.

Solution: L.H.S. $= b^2\sin 2C + c^2\sin 2B$

$= 4R^2\sin^2 B(2\sin C\cos C) + 4R^2\sin^2 C(2\sin B\cos B)$

$= 8R^2\sin B\sin C(\sin B\cos C + \cos B\sin C)$

$= 8R^2\sin B\sin C\sin(B+C)$

$= 2(2R\sin B)(2R\sin C)\sin A$

$= 2bc\sin A =$ R.H.S.

21. Problem: *Prove that* $\cot A + \cot B + \cot C = \frac{a^2+b^2+c^2}{4\Delta}$.

Solution: L.H.S. $= \Sigma\cot A = \sum\frac{\cos A}{\sin A} = \sum\left(\frac{b^2+c^2-a^2}{2bc\sin A}\right)$, (by cosine rule)

$= \sum\frac{b^2+c^2-a^2}{4\Delta}$, $\left[\because \Delta = \frac{1}{2}bc\sin A\right]$

$= \frac{1}{4\Delta}\left[b^2+c^2-a^2+c^2+a^2-b^2+a^2+b^2-c^2\right]$

$= \frac{a^2+b^2+c^2}{4\Delta} =$ R.H.S.

22. Problem: *Show that* $a\cos^2\frac{A}{2}+b\cos^2\frac{B}{2}+c\cos^2\frac{C}{2}=s+\frac{\Delta}{R}$.

Solution: L.H.S. $= \Sigma a\cos^2\frac{A}{2} = \frac{1}{2}\Sigma a(1+\cos A)$

$= \frac{1}{2}\Sigma(a+a\cos A) = \frac{1}{2}(a+b+c)+\frac{1}{2}\Sigma(2R\sin A\cos A)$

$= \frac{1}{2}(2s)+\frac{R}{2}\Sigma\sin 2A = s+\frac{R}{2}(\sin 2A+\sin 2B+\sin 2C)$

$$= s + \frac{R}{2}\left(4 \sin A \sin B \sin C\right)$$

$$= s + \frac{1}{R}\left(2R^2 \sin A \sin B \sin C\right)$$

$$= s + \frac{\Delta}{R} \quad \left(\because \Delta = 2R^2 \sin A \sin B \sin C\right) = \text{R.H.S.}$$

23. Problem: *In* ΔABC, *if* $a \cos A = b \cos B$, *prove that the triangle is either isosceles or right angled.*

Solution: $a \cos A = b \cos B \Rightarrow 2R \sin A \cos A = 2R \sin B \cos B$

$$\Rightarrow \sin 2A = \sin 2B = \sin\left(180^0 - 2B\right)$$

Hence $2A = 2B$ or $2A = 180^0 - 2B$.

$$\Rightarrow A = B \text{ or } A = (90^0 - B)$$

$$\Rightarrow a = b \text{ or } (A + B) = 90^0$$

$$\Rightarrow a = b \text{ or } C = 90^0.$$

$\therefore$ The triangle is isosceles or right angled.

24. Problem: *If* $\cot\frac{A}{2} : \cot\frac{B}{2} : \cot\frac{C}{2} = 3 : 5 : 7$, *show that* $a : b : c = 6 : 5 : 4$.

Solution: $\cot\frac{A}{2} : \cot\frac{B}{2} : \cot\frac{C}{2} = 3 : 5 : 7$

$$\Rightarrow \frac{s(s-a)}{\Delta} : \frac{s(s-b)}{\Delta} : \frac{s(s-c)}{\Delta} = 3 : 5 : 7$$

$$\Rightarrow (s-a) : (s-b) : (s-c) = 3 : 5 : 7$$

$$\Rightarrow \frac{s-a}{3} = \frac{s-b}{5} = \frac{s-c}{7} = k \text{ (say)}$$

Then $s - a = 3k,\ s - b = 5k,\ s - c = 7k$

Adding these equations, $3s - (a + b + c) = 3k + 5k + 7k = 15k$

$\Rightarrow 3s - 2s = 15k \Rightarrow s = 15k$. Hence $a = 12k, b = 10k, c = 8k$

$\therefore a : b : c = 12k : 10k : 8k = 6 : 5 : 4$.

25. Problem: *Prove that* $a^3 \cos(B - C) + b^3 \cos(C - A) + c^3 \cos(A - B) = 3abc$.

Solution: L.H.S. $= \Sigma a^3 \cos(B - C) = \Sigma a^2 (2R \sin A) \cos(B - C)$

$$= R\Sigma a^2 . \left[2\sin(B+C)\cos(B-C)\right] = R\Sigma a^2 (\sin 2B + \sin 2C)$$

$$= R\,\Sigma a^2 . (2 \sin B \cos B + 2 \sin C \cos C)$$

$$= \Sigma\left[a^2 (2R \sin B) \cos B + a^2 (2R \sin C) \cos C\right]$$

$$= \Sigma\left(a^2b\cos B + a^2c\cos C\right)$$

$$= \left(a^2b\cos B + a^2c\cos C\right) + \left(b^2c\cos C + b^2a\cos A\right) + \left(c^2a\cos A + c^2b\cos B\right)$$

$$= ab\left(a\cos B + b\cos A\right) + bc\left(b\cos C + c\cos B\right) + ca\left(c\cos A + a\cos C\right)$$

$$= ab(c) + bc(a) + ca(b) = 3\,abc = \text{R.H.S.}$$

26. Problem: *If* p_1, p_2, p_3 *are the altitudes of the* ΔABC *then, show that*

$$\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} = \frac{\cot A + \cot B + \cot C}{\Delta}.$$

Solution : Since p_1, p_2, p_3 are the altitudes of Δ ABC, we have

$$\Delta = \frac{1}{2}ap_1 = \frac{1}{2}bp_2 = \frac{1}{2}cp_3 \Rightarrow p_1 = \frac{2\Delta}{a},\ p_2 = \frac{2\Delta}{b},\ p_3 = \frac{2\Delta}{c}$$

Now $\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} = \frac{a^2+b^2+c^2}{4\Delta^2} = \frac{1}{\Delta}\left(\cot A + \cot B + \cot C\right)$

$$= \text{R.H.S.} \left[\because \cot A + \cot B + \cot C = \frac{a^2+b^2+c^2}{4\Delta}, \text{ from problem (21)}\right].$$

27. Problem: *The angle of elevation of the top point P of the vertical tower PQ of height h from a point A is* 45^0 *and from a point B is* 60^0*, where B is a point at a distance 30 meters from the point A measured along the line AB which makes an angle* 30^0 *with AQ. Find the height of the tower.*

Solution : In the Fig. 10.6

PQ = h, $\angle$PAQ $= 45^0$

$\angle$BAQ $= 30^0$ and $\angle$PBC $= 60^0$

Also AB = 30 mts.

$\therefore$ $\angle$BAP = $\angle$APB $= 15^0$.

This gives BP = AB = 30 and

h = PC + CQ = BP sin 60^0 + AB sin 30^0

$= 15\sqrt{3} + 15 = 15(\sqrt{3}+1)$ metres.

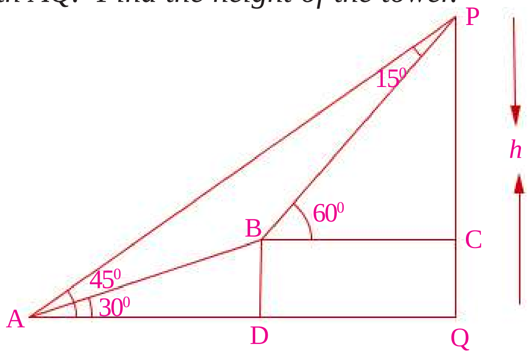


Fig. 10.6

28. Problem: *Two trees A and B are on the same side of a river. From a point C in the river the distances of the trees A and B are 250 m and 300 m respectively. If the angle C is* 45^0*, find the distance between the trees (use* $\sqrt{2} = 1.414$*).*

Solution: From the triangle ABC, using the cosine rule

$AB^2 = 250^2 + 300^2 - 2(250)(300)\cos 45^0$

$= 100(625 + 900 - 750\sqrt{2}) = 46450$

$\therefore$ AB = 215.5 m. (approximately).

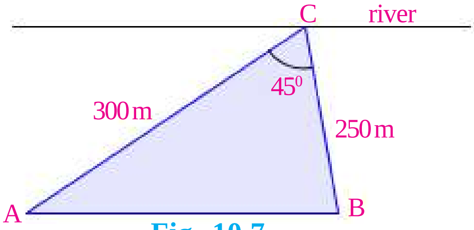


Fig. 10.7

Exercise 10(a)

(**Note** : All problems in this exercise refer to ΔABC)

I. **1.** Show that $\Sigma a(\sin B - \sin C) = 0$.

2. If $a = \sqrt{3}+1$ cms., $\lfloor B = 30^0$, $\lfloor C = 45^0$, then find c.

3. If $a = 2$ cms., $b = 3$ cms., $c = 4$ cms, then find cos A.

4. If $a = 26$ cms., $b = 30$ cms. and $\cos C = \frac{63}{65}$, then find c.

5. If the angles are in the ratio 1 : 5 : 6, then find the ratio of its sides.

6. Prove that $2(bc\cos A + ca\cos B + ab\cos C) = a^2 + b^2 + c^2$.

7. Prove that $\frac{a^2+b^2-c^2}{c^2+a^2-b^2} = \frac{\tan B}{\tan C}$.

8. Prove that $(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a+b+c$.

9. Prove that $(b - a\cos C)\sin A = a\cos A\sin C$.

10. If 4, 5 are two sides of a triangle and the included angle is 60^0, find its area.

11. Show that $b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2} = s$.

12. If $\frac{a}{\cos A} = \frac{b}{\cos B} = \frac{c}{\cos C}$, then show that ΔABC is equilateral.

II. **1.** Prove that $a\cos A + b\cos B + c\cos C = 4R\sin A\sin B\sin C$.

2. Prove that $\Sigma a^3\sin(B-C) = 0$.

3. Prove that $\frac{a\sin(B-C)}{b^2-c^2} = \frac{b\sin(C-A)}{c^2-a^2} = \frac{c\sin(A-B)}{a^2-b^2}$.

4. Prove that $\Sigma a^2\frac{\sin(B-C)}{\sin B + \sin C} = 0$.

5. Prove that $\frac{a}{bc} + \frac{\cos A}{a} = \frac{b}{ca} + \frac{\cos B}{b} = \frac{c}{ab} + \frac{\cos C}{c}$.

6. Prove that $\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B} = \frac{a^2+b^2}{a^2+c^2}$.

7. If $C = 60^0$, then show that

(i) $\frac{a}{b+c} + \frac{b}{c+a} = 1$ (ii) $\frac{b}{c^2-a^2} + \frac{a}{c^2-b^2} = 0$

8. If $a : b : c = 7 : 8 : 9$, find $\cos A : \cos B : \cos C$.

9. Show that $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$.

10. Prove that

$$(b - a)\cos C + c(\cos B - \cos A) = c \sin\left(\frac{A - B}{2}\right)\operatorname{cosec}\left(\frac{A + B}{2}\right).$$

11. Express $a \sin^2 \frac{C}{2} + c \sin^2 \frac{A}{2}$ interms of s, a, b, c.

12. If $b + c = 3a$, then find the value of $\cot\frac{B}{2} \cot\frac{C}{2}$.

13. Prove that $(b+c) \cos\frac{B+C}{2} = a\cos\frac{B-C}{2}$.

14. In a ΔABC show that $\frac{b^2 - c^2}{a^2} = \frac{\sin(B-C)}{\sin(B+C)}$.

III. 1. Prove that (i) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \frac{s^2}{\Delta}$.

(ii) $\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = \frac{bc + ca + ab - s^2}{\Delta}$.

(iii) $\frac{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}}{\cot A + \cot B + \cot C} = \frac{(a + b + c)^2}{a^2 + b^2 + c^2}$.

2. Show that (i) $\Sigma(a + b) \tan\left(\frac{A - B}{2}\right) = 0$.

(ii) $\frac{b - c}{b + c} \cot\frac{A}{2} + \frac{b + c}{b - c} \tan\frac{A}{2} = 2 \operatorname{cosec}(B - C)$.

3. (i) If $\sin\theta = \frac{a}{b + c}$, then show that $\cos\theta = \frac{2\sqrt{bc}}{b + c} \cos\frac{A}{2}$.

(ii) If $a = (b + c)\cos\theta$, then prove that $\sin\theta = \frac{2\sqrt{bc}}{b + c} \cos\frac{A}{2}$.

(iii) For any angle θ, show that $a\cos\theta = b\cos(C + \theta) + c\cos(B - \theta)$.

4. If the angles of ΔABC are in A.P. and $b : c = \sqrt{3} : \sqrt{2}$, then show that $A = 75^0$.

5. If $\frac{a^2 + b^2}{a^2 - b^2} = \frac{\sin C}{\sin(A - B)}$, prove that ΔABC is either isosceles or right angled.

6. If $\cos A + \cos B + \cos C = 3/2$, then show that the triangle is equilateral.

7. If $\cos^2 A + \cos^2 B + \cos^2 C = 1$, then show that ΔABC is right angled.

8. If $a^2 + b^2 + c^2 = 8R^2$, then prove that the triangle is right angled.

9. If $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in A.P., then prove that a, b, c are in A.P.

10. If $\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}$ are in H.P., then show that a, b, c are in H.P.

11. If $C = 90^0$ then prove that $\left(\frac{a^2+b^2}{a^2-b^2}\right)\sin(A-B) = 1$.

12. Show that $\frac{a^2}{4}\sin 2C + \frac{c^2}{4}\sin 2A = \Delta$.

13. A lamp post is situated at the middle point M of the side AC of a triangular plot ABC with BC = 7*m*, CA = 8 *m* and AB = 9 *m*. Lamp post subtends an angle 15^0 at the point B. Find the height of the lamp post.

14. Two ships leave a port at the same time. One goes 24 *km* per hour in the direction N45^0 E and other travels 32 *km* per hour in the direction S75^0 E. Find the distance between the ships at the end of 3 hours.

15. A tree stands vertically on the slant of the hill. From a point A on the ground 35 meters down the hill from the base of the tree, the angle of elevation of the top of the tree is 60^0. If the angle of elevation of the foot of the tree from A is 15^0, then find the height of the tree.

16. The upper $\frac{3}{4}$ th portion of a vertical pole subtends an angle $\mathrm{Tan}^{-1}\frac{3}{5}$ at a point in the horizontal plane through its foot and at a distance 40 m from the foot. Given that the vertical pole is at a height less than 100 m from the ground, find its height.

17. AB is a vertical pole with B at the ground level and A at the top. A man finds that the angle of elevation of the point A from a certain point C on the ground is 60^0. He moves away from the pole along the line BC to a point D such that CD = 7 *m*. From D, the angle of elevation of the point A is 45^0. Find the height of the pole.

18. Let an object be placed at some height *h* cm and let P and Q be two points of observation which are at a distance 10 *cm* apart on a line inclined at angle 15^0 to the horizontal. If the angles of elevation of the object from P and Q are 30^0 and 60^0 respectively then find *h*.

10.4 Incircle and excircles of a triangle

In this section, the notions of incircle, inradius, excircles and ex-radii are introduced.

The relations between the inradius and exradius of a circle are established.

10.4.1 Definition

The circle that touches the three sides of a triangle ΔABC internally is called the 'incircle' or 'inscribed circle' of the triangle. The centre and radius of this incircle are called incentre and inradius denoted by I and r respectively.

The point of concurrence of the internal bisectors of the angles of a triangle is the incentre I of ΔABC. I is equidistant from all sides of the triangle.

10.4.2 Theorem : *In* ΔABC,

(i) $\Delta = rs$

(ii) $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

(iii) $r = \dfrac{a}{\cot\frac{B}{2} + \cot\frac{C}{2}} = \dfrac{b}{\cot\frac{C}{2} + \cot\frac{A}{2}} = \dfrac{c}{\cot\frac{A}{2} + \cot\frac{B}{2}}$

(iv) $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, where r is the inradius.

Proof

Let I be the point of concurrence of the internal bisectors of the angles A, B, C of the ΔABC, so that I is the incentre.

Draw ID $\perp$ BC, IE $\perp$ CA, IF $\perp$ AB.

Then ID = IE = IF = r = inradius.

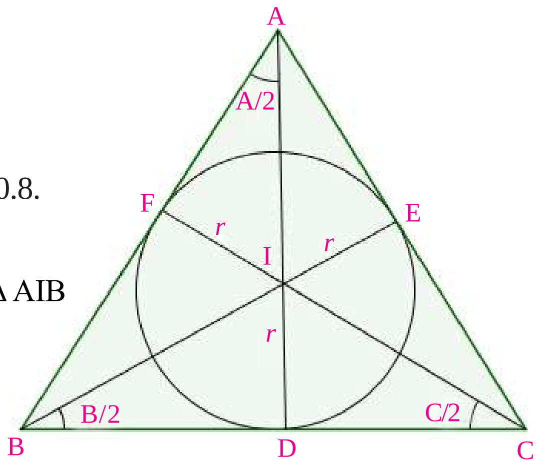


Fig. 10.8

Draw the incircle passing through D, E, F as shown in Fig. 10.8.

(i) Δ = Area of ΔABC

= Area of ΔBIC + Area of ΔCIA + Area of ΔAIB

$= \frac{1}{2}\text{BC}\cdot\text{ID} + \frac{1}{2}\text{CA}\cdot\text{IE} + \frac{1}{2}\text{AB}\cdot\text{IF}$

$= \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$

$= \frac{1}{2}r(a+b+c) = \frac{1}{2}r(2s) = rs.$

(ii) The circle passing through D, E, F is the incircle and A is an external point to the circle. AF, AE are the tangents drawn to the circle from A. Hence the length of the tangents AF and AE are equal. By a similar argument, BF = BD, CD = CE.

But AF + AE + BF + BD + CE + CD = $a + b + c$.

$\therefore 2\,\mathrm{AF} + 2\mathrm{BD} + 2\mathrm{CD} = 2s$ i.e., $2\mathrm{AF} + 2\left(\mathrm{BD} + \mathrm{CD}\right) = 2s.$

Hence $\mathrm{AF} + \mathrm{BC} = s$ i.e., $\mathrm{AF} + a = s$ i.e., $\mathrm{AF} = s - a\cdot$

In the right angled Δ AFI, $\tan\dfrac{\mathrm{A}}{2} = \dfrac{\mathrm{IF}}{\mathrm{AF}} = \dfrac{r}{s-a}$. $\therefore r = (s-a)\tan\dfrac{\mathrm{A}}{2}$.

Similarly, we can prove that $r = (s-b)\tan\dfrac{\mathrm{B}}{2}$ and $r = (s-c)\tan\dfrac{\mathrm{C}}{2}$.

(iii) From the right angled triangles IDB and IDC, we have

$$\cot\frac{\mathrm{B}}{2} = \frac{\mathrm{BD}}{r} \text{ and } \cot\frac{\mathrm{C}}{2} = \frac{\mathrm{DC}}{r} \quad \text{i.e., } \mathrm{BD} = r\cot\frac{\mathrm{B}}{2} \text{ and } \mathrm{DC} = r\cot\frac{\mathrm{C}}{2}$$

$$\therefore a = \mathrm{BD} + \mathrm{DC} = r\left(\cot\frac{\mathrm{B}}{2} + \cot\frac{\mathrm{C}}{2}\right) \Rightarrow r = \frac{a}{\cot\frac{\mathrm{B}}{2} + \cot\frac{\mathrm{C}}{2}}.$$

In a similar way, we can prove that

$$r = \frac{b}{\cot\frac{\mathrm{C}}{2} + \cot\frac{\mathrm{A}}{2}} \quad \text{and} \quad r = \frac{c}{\cot\frac{\mathrm{A}}{2} + \cot\frac{\mathrm{B}}{2}}.$$

(iv) From (iii),

$$a = r\left(\cot\frac{\mathrm{B}}{2} + \cot\frac{\mathrm{C}}{2}\right) = r\left(\frac{\cos\frac{\mathrm{B}}{2}}{\sin\frac{\mathrm{B}}{2}} + \frac{\cos\frac{\mathrm{C}}{2}}{\sin\frac{\mathrm{C}}{2}}\right)$$

$$\Rightarrow 2\mathrm{R}.\sin\mathrm{A} = r\left[\frac{\cos\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2} + \cos\frac{\mathrm{C}}{2}\sin\frac{\mathrm{B}}{2}}{\sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2}}\right]$$

$$\Rightarrow 2\mathrm{R}.2\sin\frac{\mathrm{A}}{2}\cos\frac{\mathrm{A}}{2} = r\left[\sin\left(\frac{\mathrm{B}+\mathrm{C}}{2}\right)\right] \Big/ \left(\sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2}\right)$$

$$\Rightarrow 4\mathrm{R}\sin\frac{\mathrm{A}}{2} = \frac{r}{\sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2}}; \quad \left[\because \cos\frac{\mathrm{A}}{2} = \sin\left(\frac{\mathrm{B}+\mathrm{C}}{2}\right)\right]$$

$$\Rightarrow r = 4\mathrm{R}\sin\frac{\mathrm{A}}{2}\sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2}.$$

10.4.3 Note: From Theorem 10.4.2, we have

$$\frac{\Delta}{s} = r = (s-a)\tan\frac{\mathrm{A}}{2} \quad \text{i.e., } \Delta = s(s-a)\tan\frac{\mathrm{A}}{2}$$

$$\therefore \sqrt{s(s-a)(s-b)(s-c)} = s(s-a)\tan\frac{\mathrm{A}}{2} \quad \therefore \tan\frac{\mathrm{A}}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Similarly, we can prove that

$$\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \text{ and } \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

10.4.4 Definition

The circle that touches the side BC (opposite to angle A) internally and the other two sides AB and AC externally is called the 'excircle' or 'escribed circle' opposite to the angle A.

The centre and radius of this excircle, opposite to the angle A are called excentre and exradius, denoted by I_1 *and* r_1 *respectively.*

The point of concurrence of the internal bisector of the anlge A and the external bisectors of angles B and C of ΔABC *is the excentre* I_1. I_1 *is equidistant from all sides of the triangle* (Fig. 10.9).

Similarly, we have two more excircles opposite to anlges B and C.

The centres and radii of these excircles are denoted by I_2, I_3 *and* r_2, r_3 *respectively. The triangle obtained by joining the excentres* I_1, I_2, I_3 *is called the 'extriangle'.*

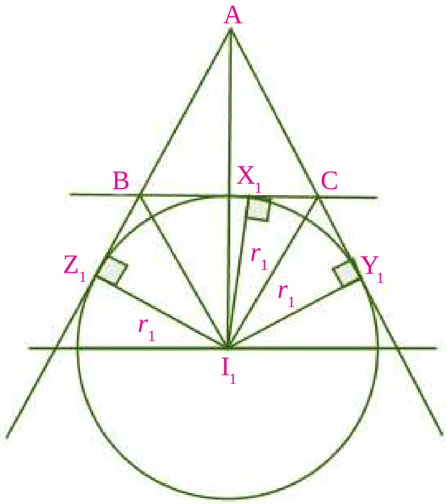


Fig. 10.9

10.4.5 Theorem : *In* ΔABC, I_1, I_2, I_3 *are excentres and* r_1, r_2 *and* r_3 *are exradii of the excircles opposite to the angles* A, B, C *respectively, then*

(i) $r_1 = \dfrac{\Delta}{s-a}, \quad r_2 = \dfrac{\Delta}{s-b}, \quad r_3 = \dfrac{\Delta}{s-c}$

(ii) $r_1 = s\tan\dfrac{A}{2} = (s-c)\cot\dfrac{B}{2} = (s-b)\cot\dfrac{c}{2}$

(iii) $r_1 = \dfrac{a}{\tan\frac{B}{2}+\tan\frac{C}{2}}, \quad r_2 = \dfrac{b}{\tan\frac{C}{2}+\tan\frac{A}{2}}, \quad r_3 = \dfrac{c}{\tan\frac{A}{2}+\tan\frac{B}{2}}$

(iv) $r_1 = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}, \quad r_2 = 4R\cos\dfrac{A}{2}\sin\dfrac{B}{2}\cos\dfrac{C}{2}$

$r_3 = 4R\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2}.$

Proof : Let I_1 be the point of concurrence of the internal bisector of angle A and external bisectors of angles B and C of the ΔABC. Then I_1 is an excentre.

Draw $I_1X_1 \perp BC$, $I_1Y_1 \perp AC$ (produced), $I_1Z_1 \perp AB$ (Produced)

Then $I_1X_1 = I_1Y_1 = I_1Z_1 = r_1 =$ ex-radius.

Draw the ex-circle passing through X_1, Y_1, Z_1 as shown in Fig. 10.9.

(i) Δ = area of ΔABC = area of ΔAI_1B + area of ΔAI_1C − area of BI_1C

$$= \frac{1}{2}AB \cdot I_1Z_1 + \frac{1}{2}AC \cdot I_1Y_1 - \frac{1}{2}BC \cdot I_1X_1 = \frac{1}{2}cr_1 + \frac{1}{2}br_1 - \frac{1}{2}ar_1$$

$$= \frac{r_1}{2}(c+b-a) = \frac{r_1}{2}\cdot 2(s-a) = r_1(s-a). \quad \therefore r_1 = \frac{\Delta}{s-a}$$

Similarly we can prove that $r_2 = \dfrac{\Delta}{s-b}$ and $r_3 = \dfrac{\Delta}{s-c}$.

(ii) Since the tangents from any external point to a circle are equal, we have $AY_1 = AZ_1$ (See Fig. 10.9). Similarly $BX_1 = BZ_1$ and $CX_1 = CY_1$.

But $BX_1 = BZ_1 = AZ_1 - AB = AY_1 - AB$; $X_1C = CY_1 = AY_1 - AC$.

Adding $BX_1 + X_1C = 2AY_1 - (AB + AC)$ i.e., $BC = a = 2AY_1 - (c+b)$

$$2AY_1 = a+b+c = 2s \quad i.e., AY_1 = s = AZ_1.$$

Then $BX_1 = s - c$ and $CX_1 = s - b$.

Now from right angled triangles AI_1Z_1, BI_1Z_1, CI_1Y_1, we have

$$\underline{|I_1AZ_1} = \frac{A}{2},\ \underline{|I_1BX_1} = 90^0 - \frac{B}{2} \text{ and } \underline{|I_1CX_1} = 90^0 - \frac{C}{2}.$$

In ΔI_1AZ_1, $\tan\dfrac{A}{2} = \dfrac{I_1Z_1}{AZ_1} = \dfrac{r_1}{s} \Rightarrow r_1 = s\tan\dfrac{A}{2}$.

In ΔI_1BX_1, $\tan\left(90^0 - \dfrac{B}{2}\right) = \dfrac{I_1X_1}{BX_1}$

$$\Rightarrow \cot\frac{B}{2} = \frac{r_1}{s-c} \Rightarrow r_1 = (s-c)\cot\frac{B}{2}.$$

In ΔI_1CX_1, $\tan\left(90^0 - \dfrac{C}{2}\right) = \dfrac{I_1X_1}{CX_1}$

$$\Rightarrow \cot\frac{C}{2} = \frac{r_1}{s-b} \Rightarrow r_1 = (s-b)\cot\frac{C}{2}.$$

$$\therefore r_1 = s\tan\frac{A}{2} = (s-c)\cot\frac{B}{2} = (s-b)\cot\frac{C}{2}.$$

Similarly we can prove that

$$r_2 = s\tan\frac{B}{2} = (s-a)\cot\frac{C}{2} = (s-c)\cot\frac{A}{2}$$

and $$r_3 = s\tan\frac{C}{2} = (s-b)\cot\frac{A}{2} = (s-a)\cot\frac{B}{2}.$$

(iii) From $\Delta\, I_1BX_1$, $\cot\left(90^0 - \frac{B}{2}\right) = \frac{BX_1}{I_1X_1}$, (from (ii))

$$\Rightarrow \tan\frac{B}{2} = \frac{BX_1}{r_1} \Rightarrow BX_1 = r_1\tan\frac{B}{2}.$$

In $\Delta\, I_1CX_1$, $\cot\left(90^0 - \frac{C}{2}\right) = \frac{CX_1}{I_1X_1} \Rightarrow \tan\frac{C}{2} = \frac{CX_1}{r_1} \Rightarrow CX_1 = r_1\tan\frac{C}{2}$

$$a = BC = BX_1 + X_1C = r_1\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right). \quad \therefore\ r_1 = \frac{a}{\tan\frac{B}{2} + \tan\frac{C}{2}}$$

Similarly, we can prove that $r_2 = \dfrac{b}{\tan\frac{C}{2} + \tan\frac{A}{2}}$ and $r_3 = \dfrac{c}{\tan\frac{A}{2} + \tan\frac{B}{2}}$.

(iv) From (iii), $a = r_1\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) = r_1\left(\dfrac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \dfrac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)$

$$\Rightarrow 2R\sin A = r_1\left(\sin\frac{B}{2}\cos\frac{C}{2} + \cos\frac{B}{2}\sin\frac{C}{2}\right)\Big/\left(\cos\frac{B}{2}\cos\frac{C}{2}\right)$$

$$\Rightarrow 2R.2\sin\frac{A}{2}\cos\frac{A}{2} = r_1\left[\sin\left(\frac{B+C}{2}\right)\right]\Big/\left(\cos\frac{B}{2}\cos\frac{C}{2}\right)$$

$$\Rightarrow 4R\sin\frac{A}{2} = \frac{r_1}{\cos\frac{B}{2}\cos\frac{C}{2}},\quad \left[\because \cos\frac{A}{2} = \sin\left(\frac{B+C}{2}\right)\right]$$

$$\Rightarrow r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Similarly, we can prove that $r_2 = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$ and

$$r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

10.4.6 Solved Problems

1. Problem: *In* Δ ABC, *prove that* $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$.

Solution: L.H.S. $= \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta}$

$$= \frac{3s - (a+b+c)}{\Delta} = \frac{3s - 2s}{\Delta} = \frac{s}{\Delta} = \frac{1}{r} = \text{R.H.S}.$$

2. Problem: *Show that* $r r_1 r_2 r_3 = \Delta^2$.

Solution: L.H.S. $= r r_1 r_2 r_3 = \frac{\Delta}{s} \cdot \frac{\Delta}{s-a} \cdot \frac{\Delta}{s-b} \cdot \frac{\Delta}{s-c}$.

$$= \frac{\Delta^4}{\Delta^2} = \Delta^2 = \text{R.H.S.}$$

3. Problem: *In an equilateral triangle, find the value of* r/R.

Solution: $\frac{r}{R} = \frac{4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}}{R} = 4\sin^3 30^0 = 4\left(\frac{1}{2}\right)^3 = \frac{1}{2}$. $\quad (\because A = B = C = 60^0)$.

4. Problem: *The perimeter of* Δ ABC *is 12 cm. and its inradius is 1 cm. Then find the area of the triangle .*

Solution: Given that $2s = 12\,cm.$ and $r = 1\,cm.$ Then $\Delta = rs = (1).(6) = 6\ sq.cm.$

5. Problem: *Show that* $r r_1 = (s-b)(s-c)$.

Solution: L.H.S. $= r r_1 = \left[(s-b)\tan\frac{B}{2}\right]\left[(s-c)\cot\frac{B}{2}\right]$

$$= (s-b)(s-c) = \text{R.H.S}.$$

6. Problem: *Express* $\frac{a\cos A + b\cos B + c\cos C}{a+b+c}$ *in terms of* R *and* r.

Solution: $\frac{2R\sin A\cos A + 2R\sin B\cos B + 2R\sin C\cos C}{a+b+c}$

$$= \frac{R(\sin 2A + \sin 2B + \sin 2C)}{2s} = \frac{R(4\sin A \,.\, \sin B \,.\, \sin C)}{2s}$$

$$= \frac{2R^2 \sin A \sin B \sin C}{sR} = \left(\frac{\Delta}{s}\right) \cdot \frac{1}{R} = \frac{r}{R}.$$

7. Problem: *In* ΔABC, $\Delta = 6$ *sq. cm. and* $s = 1.5$ *cm., find* r.

Solution: $r = \frac{\Delta}{s} = \frac{6}{1.5} = 4\,cm$.

8. Problem: *Show that* $r r_1 \cot\frac{A}{2} = \Delta$.

Solution: $r r_1 \cot\frac{A}{2} = \frac{\Delta}{s}\left(s\tan\frac{A}{2}\right)\cot\frac{A}{2} = \Delta$.

9. Problem: *If* $a = 13, b = 14, c = 15$, *find* r_1.

Solution: $2s = a + b + c = 42 \Rightarrow s = 21$. Then $s - a = 8, s - b = 7, s - c = 6$

$$\Delta^2 = 21 \times 8 \times 7 \times 6 \Rightarrow \Delta = 7 \times 12 = 84 \text{ sq.units}$$

$$\therefore r_1 = \frac{\Delta}{s-a} = \frac{84}{8} = 10.5 \text{ units.}$$

10. Problem: *If* $r\,r_2 = r_1\,r_3$, *then find B.*

Solution: $r\,r_2 = r_1\,r_3 \Rightarrow \frac{\Delta}{s} \cdot \frac{\Delta}{s-b} = \frac{\Delta}{s-a} \cdot \frac{\Delta}{s-c}$

$$\Rightarrow (s-a)(s-c) = s(s-b) \Rightarrow \frac{(s-c)(s-a)}{s(s-b)} = 1 \Rightarrow \tan^2\frac{B}{2} = 1.$$

$$\Rightarrow \tan\frac{B}{2} = 1 \Rightarrow \frac{B}{2} = 45^0 \Rightarrow B = 90^0.$$

11. Problem: *In a* ΔABC, *show that the sides* a, b, c *are in A.P. if and only if* r_1, r_2, r_3 *are in H.P.*

Solution: r_1, r_2, r_3 are in H.P. $\Leftrightarrow \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}$ are in A.P.

$$\Leftrightarrow \frac{s-a}{\Delta}, \frac{s-b}{\Delta}, \frac{s-c}{\Delta} \text{ are in A.P.} \Leftrightarrow s-a, s-b, s-c \text{ are in A.P.}$$

$$\Leftrightarrow -a, -b, -c \text{ are in A.P.} \Leftrightarrow a, b, c \text{ are in A.P.}$$

12. Problem: *If* $A = 90^0$, *show that* $2(r + R) = b + c$.

Solution: L.H.S. $= 2r + 2R = 2(s-a)\tan\frac{A}{2} + 2R.1$

$$= 2(s-a)\tan 45^0 + 2R \sin A \quad (\because A = 90^0)$$

$$= (2s - 2a).1 + a$$

$$= b + c = \text{R.H.S.}$$

13. Problem: *If* $(r_2 - r_1)(r_3 - r_1) = 2\,r_2\,r_3$, *show that* $A = 90^0$.

Solution: $(r_2 - r_1)(r_3 - r_1) = 2\,r_2\,r_3$

$$\Rightarrow \left[\frac{\Delta}{(s-b)} - \frac{\Delta}{(s-a)}\right]\left[\frac{\Delta}{(s-c)} - \frac{\Delta}{(s-a)}\right] = 2\frac{\Delta}{(s-b)} \cdot \frac{\Delta}{(s-c)}$$

$$\Rightarrow \Delta\left[\frac{s-a-s+b}{(s-b)(s-a)}\right] . \Delta\left[\frac{s-a-s+c}{(s-c)(s-a)}\right] = \frac{2\Delta^2}{(s-b)(s-c)}$$

$$\Rightarrow (b-a)(c-a) = 2(s-a)^2$$

$$\Rightarrow (b-a)(c-a) = 2\left(\frac{b+c-a}{2}\right)^2$$

$$\Rightarrow 2\left(bc - ca - ab + a^2\right) = b^2 + c^2 + a^2 + 2bc - 2ca - 2ab$$

$$\Rightarrow 2a^2 = b^2 + c^2 + a^2 \quad \Rightarrow b^2 + c^2 = a^2$$

Hence ΔABC is right angled and $A = 90^0$.

14. Problem: *Prove that* $\dfrac{r_1(r_2+r_3)}{\sqrt{r_1 r_2 + r_2 r_3 + r_3 r_1}} = a$.

Solution: $r_1 r_2 + r_2 r_3 + r_3 r_1 = \dfrac{\Delta}{s-a}\cdot\dfrac{\Delta}{s-b} + \dfrac{\Delta}{s-b}\cdot\dfrac{\Delta}{s-c} + \dfrac{\Delta}{s-c}\cdot\dfrac{\Delta}{s-a}$

$$= \Delta^2\left[\frac{s-c+s-a+s-b}{(s-a)(s-b)(s-c)}\right] = \frac{\Delta^2(3s-2s)s}{s(s-a)(s-b)(s-c)} = \frac{\Delta^2 s^2}{\Delta^2} = s^2$$

and $r_1(r_2+r_3) = \dfrac{\Delta}{s-a}\left[\dfrac{\Delta}{s-b} + \dfrac{\Delta}{s-c}\right] = \dfrac{\Delta^2}{s-a}\left[\dfrac{s-c+s-b}{(s-b)(s-c)}\right]$

$$= \frac{s.\Delta^2 a}{s(s-a)(s-b)(s-c)} = \frac{s.\Delta^2 a}{\Delta^2} = as$$

Hence $\dfrac{r_1(r_2+r_3)}{\sqrt{r_1 r_2 + r_2 r_3 + r_3 r_1}} = \dfrac{as}{\sqrt{s^2}} = a$.

15. Problem: *Show that* $\dfrac{1}{r^2} + \dfrac{1}{r_1^2} + \dfrac{1}{r_2^2} + \dfrac{1}{r_3^2} = \dfrac{a^2+b^2+c^2}{\Delta^2}$.

Solution: L.H.S. $= \dfrac{1}{r^2} + \dfrac{1}{r_1^2} + \dfrac{1}{r_2^2} + \dfrac{1}{r_3^2}$

$$= \frac{s^2}{\Delta^2} + \frac{(s-a)^2}{\Delta^2} + \frac{(s-b)^2}{\Delta^2} + \frac{(s-c)^2}{\Delta^2}$$

$$= \frac{1}{\Delta^2}\left[s^2 + (s-a)^2 + (s-b)^2 + (s-c)^2\right]$$

$$= \frac{1}{\Delta^2}\left[s^2 + s^2 - 2as + a^2 + s^2 - 2bs + b^2 + s^2 - 2cs + s^2\right]$$

$$= \frac{1}{\Delta^2}\left[4s^2 - 2s(a+b+c) + a^2 + b^2 + c^2\right]$$

$$= \frac{1}{\Delta^2}\left[4s^2 - 2s(2s)\right] + \frac{a^2+b^2+c^2}{\Delta^2}$$

$$= \frac{a^2+b^2+c^2}{\Delta^2} = \text{R.H.S.}$$

16. Problem: *Prove that* $\Sigma\left(r+r_1\right)\tan\left(\frac{B-C}{2}\right)=0$.

Solution: $r + r_1 = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} + 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$$= 4R\sin\frac{A}{2}\left[\sin\frac{B}{2}\sin\frac{C}{2}+\cos\frac{B}{2}\cos\frac{C}{2}\right]$$

$$= 4R\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right).$$

$$\Rightarrow \left(r+r_1\right)\tan\left(\frac{B-C}{2}\right) = 4R\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right)\left[\frac{\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B-C}{2}\right)}\right]$$

$$= 4R\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)$$

$$= 2R\left(\sin B-\sin C\right) = b-c.$$

Hence $\Sigma\left(r+r_1\right)\tan\left(\frac{B-C}{2}\right)=\Sigma\left(b-c\right)=0.$

17. Problem: *Show that* $\frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab}=\frac{1}{r}-\frac{1}{2R}$.

Solution: L.H.S. $= \frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab} = \frac{1}{abc}\left[ar_1+br_2+cr_3\right]$

$$= \frac{1}{abc}\left[\Sigma a\,.\,s\tan\frac{A}{2}\right] = \frac{s}{abc}\Sigma 2R\sin A\tan\frac{A}{2}$$

$$= \frac{s}{abc}\Sigma\left[2R\,.\,2\sin\frac{A}{2}\cos\frac{A}{2}\,.\left(\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}\right)\right]$$

$$= 4\frac{Rs}{abc}\Sigma\left(\sin^2\frac{A}{2}\right)=\frac{s}{\Delta}\Sigma\left(\frac{1-\cos A}{2}\right),\quad\left(\because \Delta=\frac{abc}{4R}\right)$$

$$= \frac{1}{2r}\left(1-\cos A+1-\cos B+1-\cos C\right)\quad\left(\because r=\Delta/s\right)$$

$$= \frac{1}{2r}\left[3-\left(\cos A+\cos B+\cos C\right)\right]$$

$$= \frac{1}{2r}\left[3-\left(1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\right]$$

$$= \frac{1}{2r}\left[2-\left(\frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R}\right)\right]$$

$$= \frac{1}{2r}\left[2-\frac{r}{R}\right] = \frac{1}{r}-\frac{1}{2R} = \text{R.H.S.}$$

18. Problem: *If $r : R : r_1 = 2:5:12$, then prove that the triangle is right angled at A.*

Solution: If $r : R : r_1 = 2:5:12$, then $r = 2k$, $R = 5k$, and $r_1 = 12k$ for some k.

$$r_1 - r = 12k - 2k = 10k = 2(5k) = 2R$$

$$\Rightarrow 4R\sin\frac{A}{2}\left[\cos\frac{B}{2}\cos\frac{C}{2}-\sin\frac{B}{2}\sin\frac{C}{2}\right] = 2R$$

$$\Rightarrow 2\sin\frac{A}{2}\cos\left(\frac{B+C}{2}\right) = 1$$

$$\Rightarrow \sin^2\frac{A}{2} = \frac{1}{2}, \qquad \left[\because \cos\left(\frac{B+C}{2}\right) = \sin\frac{A}{2}\right]$$

$$\Rightarrow \sin\frac{A}{2} = \frac{1}{\sqrt{2}} = \sin 45^0 \Rightarrow \frac{A}{2} = 45^0 \Rightarrow A = 90^0.$$

Hence the triangle is right angled at A.

19. Problem: *Show that* $r + r_3 + r_1 - r_2 = 4R\cos B$.

Solution: $r + r_3 = 4R\sin\frac{C}{2}\left[\sin\frac{A}{2}\sin\frac{B}{2}+\cos\frac{A}{2}\cos\frac{B}{2}\right] = 4R\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right).$

$$r_1 - r_2 = 4R\cos\frac{C}{2}\left[\sin\frac{A}{2}\cos\frac{B}{2}-\cos\frac{A}{2}\sin\frac{B}{2}\right]$$

$$= 4R\cos\frac{C}{2}\sin\left(\frac{A-B}{2}\right).$$

$$\therefore r + r_3 + r_1 - r_2 = 4R\left[\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)+\cos\frac{C}{2}\sin\left(\frac{A-B}{2}\right)\right]$$

$$= 4R\sin\left(\frac{C}{2}+\frac{A}{2}-\frac{B}{2}\right)$$

$$= 4R\sin\left(90^0-\frac{B}{2}-\frac{B}{2}\right) = 4R\cos B.$$

20. Problem: *If* A, A_1, A_2, A_3 *are the areas of incircle and excircles of a triangle respectively, then prove that* $\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}} = \frac{1}{\sqrt{A}}$.

Solution: If r, r_1, r_2, r_3 are the inradius and exradii of the circles whose areas are A, A_1, A_2, A_3 respectively, then $A = \pi r^2$, $A_1 = \pi r_1^2$, $A_2 = \pi r_2^2$, $A_3 = \pi r_3^2$.

$$\sqrt{A} = \sqrt{\pi}\,.r,\ \sqrt{A_1} = \sqrt{\pi}\,.r_1,\ \sqrt{A_2} = \sqrt{\pi}\,.r_2,\ \sqrt{A_3} = \sqrt{\pi}\,.r_3.$$

$$\therefore \frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}} = \frac{1}{\sqrt{\pi}}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{r}\right) = \frac{1}{\sqrt{A}}.$$

21. Problem: *Show that* $(r_1 + r_2)\sec^2\frac{C}{2} = (r_2 + r_3)\sec^2\frac{A}{2} = (r_3 + r_1)\sec^2\frac{B}{2}$.

Solution: $r_1 + r_2 = 4R\cos\frac{C}{2}\left[\sin\frac{A}{2}\cos\frac{B}{2} + \cos\frac{A}{2}\sin\frac{B}{2}\right]$

$$= 4R\cos\frac{C}{2}\sin\left(\frac{A+B}{2}\right) = 4R\cos^2\frac{C}{2}$$

$$\Rightarrow (r_1 + r_2)\sec^2\frac{C}{2} = 4R.$$

Similarly, we can show that $(r_2 + r_3)\sec^2\frac{A}{2} = (r_3 + r_1)\sec^2\frac{B}{2} = 4R$.

$$\therefore (r_1 + r_2)\sec^2\frac{C}{2} = (r_2 + r_3)\sec^2\frac{A}{2} = (r_3 + r_1)\sec^2\frac{B}{2}.$$

22. Problem: *In* ΔABC, *if* AD, BE, CF *are the perpendiculars drawn from the vertices* A, B, C *to the opposite sides, show that*

$$\text{(i)}\ \frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF} = \frac{1}{r} \quad \textit{and} \quad \text{(ii)}\ AD\,.\,BE\,.\,CF = \frac{(abc)^2}{8R^3}.$$

Solution: Since $AD \perp BC$, (see Fig. 10.10),

$$\Delta = \frac{1}{2}\ BC\,.\,AD \Rightarrow AD = \frac{2\Delta}{BC} = \frac{2\Delta}{a}.$$

Similarly we get that $BE = \frac{2\Delta}{b}$ and $CF = \frac{2\Delta}{c}$.

Now (i) $\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF} = \frac{a}{2\Delta} + \frac{b}{2\Delta} + \frac{c}{2\Delta} = \frac{2s}{2\Delta} = \frac{s}{\Delta} = \frac{1}{r}$.

(ii) $AD\,.\,BE\,.\,CF = \frac{2\Delta}{a}\cdot\frac{2\Delta}{b}\cdot\frac{2\Delta}{c} = \frac{8\Delta^3}{abc}$

$$= \frac{8}{abc}\left(\frac{abc}{4R}\right)^3 = \frac{(abc)^2}{8R^3}.$$

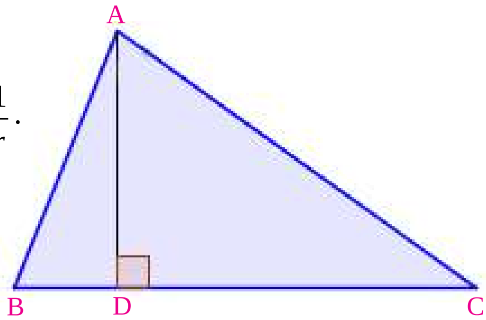


Fig. 10.10

23. Problem: *In* ΔABC, *if* $r_1 = 8, r_2 = 12, r_3 = 24$, *find* a, b, c.

Solution: $r_1 = 8 \Rightarrow \dfrac{\Delta}{s-a} = 8 \Rightarrow s - a = \dfrac{\Delta}{8}$

$r_2 = 12 \Rightarrow \dfrac{\Delta}{s-b} = 12 \Rightarrow s - b = \dfrac{\Delta}{12}$.

$r_3 = 24 \Rightarrow \dfrac{\Delta}{s-c} = 24 \Rightarrow s - c = \dfrac{\Delta}{24}$.

Adding, $3s - (a+b+c) = \Delta\left(\dfrac{1}{8} + \dfrac{1}{12} + \dfrac{1}{24}\right) \Rightarrow 3s - 2s = \Delta\left(\dfrac{1}{4}\right) \Rightarrow s = \dfrac{\Delta}{4}$.

Now $r = \dfrac{\Delta}{s} = \dfrac{\Delta}{\left(\dfrac{\Delta}{4}\right)} = 4$.

But $\Delta^2 = r\, r_1 r_2 r_3 = 4 \times 8 \times 12 \times 24 = (8 \times 12)^2 \Rightarrow \Delta = 96$ sq. units.

and $s = \dfrac{\Delta}{r} = \dfrac{96}{4} = 24$. Hence $a = s - \dfrac{\Delta}{r_1} = 24 - \dfrac{96}{8} = 24 - 12 = 12$.

Similarly $b = 24 - \dfrac{96}{12} = 24 - 8 = 16$; $c = 24 - \dfrac{96}{24} = 24 - 4 = 20$.

24. Problem: *Show that* $\dfrac{ab - r_1 r_2}{r_3} = \dfrac{bc - r_2 r_3}{r_1} = \dfrac{ca - r_3 r_1}{r_2}$.

Solution: $ab - r_1 r_2 = (2R\sin A)(2R\sin B) - \left(4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}\right)\left(4R\cos\dfrac{A}{2}\sin\dfrac{B}{2}\cos\dfrac{C}{2}\right)$

$= 4R^2\sin A\sin B - 4R^2\left(\cos^2\dfrac{C}{2}\right)\left(2\sin\dfrac{A}{2}\cos\dfrac{A}{2}\right)\left(2\sin\dfrac{B}{2}\cos\dfrac{B}{2}\right)$

$= 4R^2\sin A\sin B - 4R^2\cos^2\dfrac{C}{2}\sin A\sin B$

$= 4R^2\sin A\sin B\left(1 - \cos^2\dfrac{C}{2}\right) \quad = 4R^2\sin A\sin B\sin^2\dfrac{C}{2}$.

Now $\dfrac{ab - r_1 r_2}{r_3} = \dfrac{4R^2\left(2\sin\dfrac{A}{2}\cos\dfrac{A}{2}\right)\left(2\sin\dfrac{B}{2}\cos\dfrac{B}{2}\right)\sin^2\dfrac{C}{2}}{4R\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2}}$.

$= 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} = r$.

Similarly we can show that $\dfrac{bc - r_2 r_3}{r_1} = \dfrac{ca - r_3 r_1}{r_2} = r$.

Exercise 10(b)

(**Note:** All problems in this exercise have reference to Δ ABC)

I. 1. Express $\Sigma r_1 \cot\frac{A}{2}$ interms of s.

2. Show that $\Sigma a \cot A = 2(R + r)$.

3. In ΔABC, prove that $r_1 + r_2 + r_3 - r = 4R$.

4. In ΔABC, prove that $r + r_1 + r_2 - r_3 = 4R\cos C$.

5. If $r + r_1 + r_2 = r_3$, then show that $C = 90^0$.

II. 1. Prove that $4(r_1 r_2 + r_2 r_3 + r_3 r_1) = (a + b + c)^2$.

2. Prove that $\left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right)\left(\frac{1}{r} - \frac{1}{r_3}\right) = \frac{abc}{\Delta^3} = \frac{4R}{r^2 s^2}$.

3. Prove that $r(r_1 + r_2 + r_3) = ab + bc + ca - s^2$.

4. Show that $\Sigma \frac{r_1}{(s-b)(s-c)} = \frac{3}{r}$.

5. Show that $(r_1 + r_2)\tan\frac{C}{2} = (r_3 - r)\cot\frac{C}{2} = c$.

6. Show that $r_1 r_2 r_3 = r^3 \cot^2\frac{A}{2}\cot^2\frac{B}{2}\cot^2\frac{C}{2}$.

III. 1. Show that $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$.

2. Show that $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2 + \frac{r}{2R}$.

3. Show that $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = 1 - \frac{r}{2R}$.

4. Show that (i) $a = (r_2 + r_3)\sqrt{\frac{r\, r_1}{r_2\, r_3}}$ (ii) $\Delta = r_1 r_2\sqrt{\frac{4R - r_1 - r_2}{r_1 + r_2}}$.

5. Prove that $r_1^2 + r_2^2 + r_3^2 + r^2 = 16R^2 - (a^2 + b^2 + c^2)$.

6. If p_1, p_2, p_3 are altitudes drawn from vertices A, B, C to the opposite sides of a triangle respectively, then show that

(i) $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=\frac{1}{r}$ (ii) $\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}=\frac{1}{r_3}$

(iii) $p_1\,p_2\,p_3=\frac{(abc)^2}{8R^3}=\frac{8\Delta^3}{abc}$

7. If $a=13, b=14, c=15$, show that $R=\frac{65}{8}, r=4, r_1=\frac{21}{2}, r_2=12$ and $r_3=14$.

8. If $r_1=2, r_2=3, r_3=6$ and $r=1$, Prove that $a=3, b=4$ and $c=5$.

Key Concepts

- **Law of sines or sine rule :** In ΔABC, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$

- **Cosine rule or Law of cosines :** In ΔABC, $b^2=c^2+a^2-2ca\cos B$;

  $c^2=a^2+b^2-2ab\cos C$; $a^2=b^2+c^2-2bc\cos A$

- **Napier analogy or tangent rules :** In ΔABC, $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$

  $\tan\frac{C-A}{2}=\frac{c-a}{c+a}\cot\frac{B}{2}$; $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$

- **Half - angle formulae:** In ΔABC,

  $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$, $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$ and

  $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ and similar expressions for

  $\sin\frac{B}{2}, \sin\frac{C}{2}$; $\cos\frac{B}{2}, \cos\frac{C}{2}$, and $\tan\frac{B}{2}, \tan\frac{C}{2}$.

- Δ = Area of the triangle ABC $=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\frac{1}{2}ab\sin C$

  $\Delta=\sqrt{s(s-a)(s-b)(s-c)}=\frac{abc}{4R}$.

- In Δ ABC,

$$\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}; \quad \frac{b+c}{a} = \frac{\cos\left(\frac{B-C}{2}\right)}{\sin\frac{A}{2}}; \quad \frac{c+a}{b} = \frac{\cos\left(\frac{C-A}{2}\right)}{\sin\frac{B}{2}}.$$

- The circle that touches the three sides of a Δ ABC internally is called the **incircle** or **inscribed circle** of the triangle. The centre and radius of this incircle are called incentre and inradius, denoted by I and r respectively.
- The point of concurrence of the internal bisectors of a triangle is called the **incentre** I of Δ ABC.
- The circle that touches the sides BC internally and the other two sides AB and AC externally, is called the **excircle** or **escribed circle** opposite to angle A.
- The centre and radius of the circle opposite to angle A are called excentre and exradius denoted by I_1 and r_1 respectively. Similarly r_2, r_3 and I_2, I_3.
- In Δ ABC, $r = \frac{\Delta}{s}$, $r_1 = \frac{\Delta}{s-a}$, $r_2 = \frac{\Delta}{s-b}$, $r_3 = \frac{\Delta}{s-c}$

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}; \qquad r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2};$$

$$r_2 = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}; \qquad r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

Historical Note

Triangles have been used in decorative pattern from the earliest times. Some of the first geometrical discoveries related to the properties of triangles, like the one about right angled triangle, are traditionally ascribed to *Pythagoras*. But the *sulbasutra* period has given us, in India, several interesting properties of triangles, centuries before *Pythagoras.* Like Ceva's theorem, many theorems deal with properties of triangles.

But the systematic approach to Geometry in general and properties of triangles in particular can be traced to Greek period and to *Euclid's* Elements. The *Greeks* insisted that geometric fact must be established, not by empirical procedures, as was the practice in many earlier cultures, but by deductive reasoning and geometrical conclusions must be arrived at by logical demonstration rather than by trial and error experimentation.

In short, the Greeks transformed the empirical geometry of the ancient *Egyptians, Babylonians* and *Indians*, into what we might call, deductive or demonstrative geometry.

Answers

Exercise 10(a)

I. **2.** 2 cms. **3.** 7/8 **4.** 8

5. $\sqrt{3}-1 : \sqrt{3}+1 : 2\sqrt{2}$ **10.** $5\sqrt{3}$ sq. units

II. **8.** 14 : 11 : 6 **11.** $(s - b)$ **12.** 2

III. **13.** $7(2-\sqrt{3})\,m$ **14.** 86.4 km(approx.) **15.** $35\sqrt{2}\,m$

16. 40 m **17.** $\frac{7\sqrt{3}}{2}(\sqrt{3}+1)$ **18.** $5\sqrt{2}$

Exercise10(b)

I. **1.** $3\,s$

Appendix

- ☞ Sets
- ☞ Relations
- ☞ Sequences and Series
- ☞ Mathematical Reasoning

No Question is to be set in the IPE, Mathematics - IA from the topics mentioned above

Chapter 1

Sets

Introduction

Set theory owes its origin to the German mathematician George Cantor (1845 - 1918) who developed this theory while he was working on trigonometric series. The concept of a set is fundamental for the development of abstract algebra. Knowledge of sets is heavily required in the study of several branches of mathematics like Analysis, Probability, Number Theory, Discrete Mathematics, Graph Theory etc.

1.1 Set

A **set** is a well defined collection of objects. By well definedness we mean that it is possible to decide whether a given object does belong to the given collection or not.

Obejcts of a set are called elements.

1.2 Examples

The following collections constitute a set :

1. The vowels in the English alphabet : a, e, i, o, u constitute a set.
2. All natural numbers which are divisors of 36.
3. All prime numbers.
4. All the rivers flowing in India.
5. All rational numbers.

Note: Elements of a set are represented generally by lower case letters a, b, c, p, q, r, ... and sets by upper case (capital) letters A, B, C, L, M, N, ...

1.3 Representation of a set

There are two methods of representing a set. They are

(i) Roster or tabular form, and

(ii) Set builder form

In the **Roster form**, elements of the set are listed in a row, seperated by commas and enclosed within a brace { }. For example the set of all divisors of 36 is represented in the roster form as : {1, 2, 3, 4, 6, 9, 12, 18, 36}. Though it is customary to list the elements in an order, ordering of elements has no importance or relevance in the listing. Similarly all elements in the listing are taken to be distinct.

In the **Set builder form**, a common property or a characteristic property that is possessed by all the elements in the well defined collection is used to describe that set. A general element, say x is chosen to represent the set, followed by a colon (:) or a vertical line (|) which is then followed by the characteristic property satisfied by all those elements and enclose the whole description in braces. For example :

Q : $\{x \mid x$ is a rational number$\}$.

In the above description, the braces stand for 'the set of all' and the vertical line for 'such that'. We read it as "**Q** is the set of all x such that x is a rational number".

1.4 Classification (Types) of sets

1. Empty set or null set

There is a unique set that does not have any element in it. This set having no elements is called the **null set** or **empty set**. It is usually denoted by ϕ or { }.

Example : $A = \{x : x^2 = 16$ and x is an odd integer$\}$ is an empty set, because $x^2 = 16$ is not satisfied by any odd integer.

If A is not the empty set, then we say that A is a **non-empty set**.

2. Finite and Infinite sets

From the examples (1, 2), we understand that the number of elements in the set may be finite or infinite. A set is said to be finite if it consists of definite number of elements. Otherwise it is infinite.

3. Equality of sets

If A and B are two sets such that every member of A is a member of B and every member of B is a member of A, then we say that A and B are equal and write A = B. For example,

Let A = {1, 2, 3} and B = {2, 3, 1}. Then A = B.

4. Subset and Superset

If A and B are any two sets, we say that A is a subset of B or B is a superset of A if $x \in A \Rightarrow x \in B$ and we write $A \subseteq B$. If B contains all the elements of A and at least one element which is not in A i.e.,

$A \subseteq B$ and $A \neq B$, then we say that A is a proper subset of B and we write $A \subset B$ or $B \supset A$. Every non-empty set has two improper subsets namely the empty set and the set itself.

Example : The set **Q** of rational numbers is a proper subset of the set of real numbers **R**.

5. **Power set of a set :** If A is any set, then the set of all subsets of A is called the **power set** of A and is denoted by $\rho(A)$ or $\mathscr{P}(A)$.

   Example : Let $A = \{1, 2, 3\}$. Then

   $$\rho(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}.$$

6. **Universal set :** It is customary that the superset A, taken here to define the complement, is called the Universal set and is denoted by U. The complement of a set S in the universal set U consists of all elements of U which are not the elements of S. It is denoted by $S' = U \setminus S$. Observe here that $(A')' = A$ for any subset A of the universal set U.

7. **Disjoint sets :** If two sets A and B are such that they do not have any elements in common, i.e., $A \cap B = \phi$, then A and B are said to be **disjoint sets**.

   Example : Let $A = \{x \mid x$ is a vowel in English alphabet$\}$, and

   $B = \{y \mid y$ is a consonant in English alphabet$\}$.

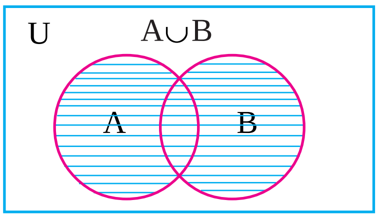
   Then $A \cap B = \phi$ and hence A, B are disjoint sets.

1.5 Venn Diagram

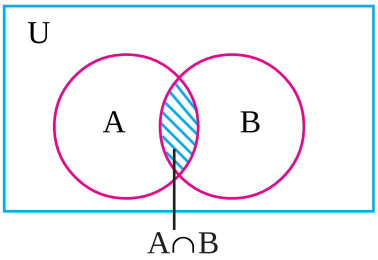
Sets, relationships between sets and operations on sets can be more conveniently represented by diagrams, known as '**Venn diagrams**', named after the English logician John Venn (1834-1883). In these diagrams, a universal set is represented by a rectangle and all its subsets by small circles, inside the universal set.

Examples

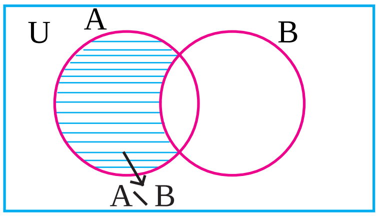
1. The union of two sets A and B i.e., $A \cup B = \{x \mid x \in A$ or $x \in B\}$ can be represented by the shaded portion in the following Venn diagram.



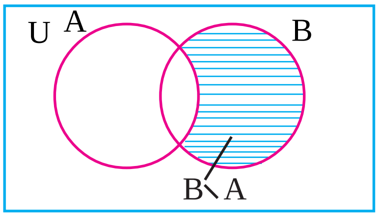
2. The intersection of two sets A and B i.e., $A \cap B = \{x \mid x \in A$ and $x \in B\}$ can be represented by the shaded portion in the following Venn diagram.



3. The difference of the sets A and B i.e., $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$ can be represented by the shaded portion of the following Venn diagram.



4. Similarly $B \setminus A = \{x \mid x \in B \text{ and } x \notin A\}$ is shown by the shaded portion of the following Venn diagram.



Note: From these Venn diagrams we observe that the sets $A \cap B$, $A \setminus B$ and $B \setminus A$ are mutually disjoint i.e., the intersection of any two of them is the empty set ϕ.

1.6 Operations on sets

Let A and B be any two sets. Then we define the following operations on these sets.

1. The **union** of A and B, denoted by $A \cup B$ is defined as $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
2. The **intersection** of A and B, denoted by $A \cap B$ is defined as $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.
3. The **difference** of the sets A and B, denoted by $A \setminus B$ is defined as $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$. $A \setminus B$ is also referred to as the complement of B with respect to A.
4. The **symmetric difference** of A and B, denoted by $A \Delta B$, is defined as $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

1.7 Example

Let A = {2, 4, 6, 8, 10, 12}, B = {2, 6, 8, 12, 15, 18}.

Then $A \cup B = \{2, 4, 6, 8, 10, 12, 15, 18\}$

$A \cap B = \{2, 6, 8, 12\}$

$A \setminus B = \{4, 10\}$

$A \Delta B = \{4, 10\} \cup \{15, 18\} = \{4, 10, 15, 18\}$.

1.8 Complement of a set

Let A be any set and $B \subset A$. The complement of B in A, denoted by B' is defined as the set $B' = \{ x \in A : x \notin B \}$

we observe that $B' = A \setminus B$, since $B \subset A$.

Example : Let $A = \{x \,|\, x$ is an alphabet in English$\}$, and

$B = \{y \,|\, y$ is a vowel in the English alphabet$\}$.

Then B' in A is : $\{z \,|\, z$ is a consonant in the English alphabet$\}$.

1.9 Some Properties of operations on sets

1. Let A, B, C be three sets. The following properties, satisfied by the operations of union, intersection and complement of sets, can be verified either from their definition or by the Venn diagrams.

(i) $A \cup A = A$	$A \cap A = A$	Idempotent Laws
(ii) $A \cup B = B \cup A$	$A \cap B = B \cap A$	Commutative Laws
(iii) $(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$	Associative Laws
(iv) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
(v) If A, B are subsets of U, then $(A \cup B)' = A' \cap B'$	$(A \cap B)' = A' \cup B'$	De'Morgan's Laws
(vi) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$	$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$	
(vii) $A \cup A' = U$	$A \cap A' = \phi$	
(viii) $(A')' = A$, $U' = \phi$, $\phi' = U$		

2. The following properties can also be verified from definitions.

If A and B are two sets, then

(i) $A \subset B \Leftrightarrow A \cup B = B \Leftrightarrow A \cap B = A$.

(ii) $A \subseteq B$ and $B \subseteq A \Leftrightarrow A = B$.

(iii) $A \setminus B = A \setminus (A \cap B) = (A \cup B) \setminus B$.

(iv) $A \setminus (A \setminus B) = A \cap B$.

(v) $(A \setminus B) \cup (A \cap B) = A$.

(vi) $(A \setminus B) \cup (B \setminus A) \cup (A \cap B) = A \cup B$.

(vii) $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

(viii) If B and C are subsets of A, then

$B \setminus C = B \cap C'$, $B \setminus C' = B \cap C$.

3. The following properties follow from the symmetric difference of sets.

If A, B, C are three sets, then

(i) $A \Delta \phi = A$

(ii) $A \Delta A = \phi$

(iii) $A \Delta B = A \Delta C \Rightarrow B = C$

(iv) $A \Delta B = B \Delta A$

(v) $A \Delta (B \Delta C) = (A \Delta B) \Delta C$.

We shall now establish the result (v) :

we have from 2(vii) that $A \Delta B = (A \cup B) \cap (A' \cup B')$.

$$\Rightarrow (A \Delta B)' = (A \cap B) \cup (A' \cap B'), \text{ by De'Morgan's laws.}$$

Hence $(A \Delta B) \Delta C = \{(A \Delta B) \cup C\} \cap \{(A \Delta B)' \cup C'\}$

$$= [(A \cup B) \cap (A' \cup B') \cup C] \cap [(A \cap B) \cup (A' \cap B') \cup C']$$

$$= (A \cup B \cup C) \cap (A' \cup B' \cup C) \cap [\{A \cup (A' \cap B')\} \cup C'] \cap [\{B \cup (A' \cap B')\} \cup C']$$

$$= (A \cup B \cup C) \cap (A' \cup B' \cup C) \cap (A \cup B' \cup C') \cap (A' \cup B \cup C').$$

Since the expression on the right of the above equation is unchanged by the interchange of A and C, we have

$$(A \Delta B) \Delta C = (C \Delta B) \Delta A = A \Delta (C \Delta B) = A \Delta (B \Delta C).$$

1.10 Cardinal number of a set

1.10.1 Definition

If A is a finite set, then the number of distinct elements of the set A is called the cardinal number of that set and is denoted by n(A).

The following properties can be verified in terms of the cardinal number of sets.

Let A, B, C be finite sets. Then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B) = n(A) + n(B)$, if A and B are disjoint sets

(ii) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$.

(iii) $n(A \setminus B) = n(A) \setminus n(A \cap B)$.

Chapter 2

Relations

Introduction

The word **relation** connotes a recognizable connection between two elements or entities. If a relation R is associated between two sets A and B, then we talk of a relation from A to B. If $a \in$ A and $b \in$ B, then if a is related to b, we write a R b and the ordered pair $(a, b) \in$ R.

2.1 Cartesian Product of sets

2.1.1 Definition

Let A and B be two non-empty sets. Then the set of all ordered pairs (x, y) where $x \in A$ and $y \in B$ is called the cartesian product of A and B and is denoted by $A \times B$.

We write $A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$.

Example : Let $A = \{a, b, c\}$, $B = \{p, q\}$.

Then $A \times B = \{(a, p), (b, p), (c, p), (a, q), (b, q), (c, q)\}$

and $B \times A = \{(p, a), (p, b), (p, c), (q, a), (q, b), (q, c)\}$.

2.1.2 Observations

(i) $A \times B \neq B \times A$, since for any distinct x and y, $(x, y) \neq (y, x)$.

(ii) $A \times B = B \times A \Leftrightarrow A = B$.

(iii) If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$.

2.2 Relation on sets

2.2.1 Definition : Relation

A relation R from a non-empty set A to a non-empty set B is defined as a subset of the cartesian product of A and B. i.e., $R \subseteq A \times B$.

If R is a relation from A to B and if $(a, b) \in R$, *then we also write aRb.*

2.2.2 Definition : Domain and Range of a relation

If A and B are two sets and if R is a relation from A to B, then

(i) $\{x \in A \mid (x, y) \in R$ *for some* $y \in B\}$ *is called the* ***domain*** *of R.*

(ii) $\{y \in B \mid (x, y) \in R$ *for some* $x \in A\}$ *is called the* ***range*** *of R.*

2.2.3 Types of relations : Let A be a set and R be a relation on A.

Then (i) For any set A, a relation from A to A is called a **binary relation** on A.

(ii) For all $a \in A$, if $(a, a) \in R$, then R is called a **reflexive relation** on A.

(iii) For any $a, b \in A$, if $(a, b) \in R \Rightarrow (b, a) \in R$ then R is called a **symmetric relation** on A.

(iv) For any $a, b, c \in A$ if $(a, b) \in R$, $(b, c) \in R \Rightarrow (a, c) \in R$, then R is called a **transitive relation** on A.

2.2.4 Definition : Equivalence relation

Any relation R on a non-empty set A which is reflexive, symmetric and transitive is called ***an equivalence relation*** *on A.*

2.2.5 Note : (i) A consequence of an equivalence relation on a set is that, it separates into the elements of that set into a union of pairwise disjoint subsets whose collection is called the **partition** of the set. Each disjoint subset is called an **equivalence class**.

Chapter 3

Sequences and Series

Introduction

'Sequences and Series' play a significant role in Mathematics. We will learn about convergence and divergence of sequences and series through various mathematical tests or procedures in higher classes. However in this chapter, we recollect some basic concepts and properties of sequences and series which are in A.P and G.P.

3.1 Definition

A sequence is a function from the set of natural numbers **N** into **R**.

Suppose $f : \mathbf{N} \to A$. Then, the function f is written as $\{f(1), f(2), f(3), \ldots\}$.

Therefore $\{f(n) : n \in \mathbf{N}\}$ is the range of the sequence.

$\{f(n)\}$ denotes the elements of the sequence and the elements are usually written as $\{a_n\}$. a_n is called the n^{th} (general) term of the sequence $\{a_n\}$.

- The following are examples of sequences.

  (i) 2, 4, 8, 16, 32, ...

  general term = 2^n, where n is a positive integer.

  (ii) 3, 6, 9, 12, 15, ...

  general term = $3n$.

  (iii) 2, 3, 5, 7, 11, ... is the sequence of prime numbers.

3.2 Series

An expression of the form

$$a_1 + a_2 + ... + a_n + ...$$

where $a_1, a_2, ..., a_n, ...$ are real numbers is called a *series* or an infinite series.

In higher mathematics, we attach a real number s called the sum of the series, in some cases. This s is related to the sequence $\{s_n\}$ where

$$s_n = a_1 + a_2 + ... + a_n.$$

However, in this chapter we confine only to finite series, i.e., $a_1 + a_2 + ... + a_n$ (sum of a finite number of real numbers).

- The infinite series $a_1 + a_2 + ... + a_n + ...$ is also denoted by $\sum_{n=1}^{\infty} a_n$.

  and the finite series $a_1 + a_2 + ... + a_n$ is denoted by $\sum_{k=1}^{n} a_k$.
- There are various types of sequences. Prominent among them are (i) Arithmetic sequence and (ii) Geometric sequence.
- Arithmetic series or Arithmetic progression and Geometric series or Geometric Progression arise from these sequences.

3.2.1 Arithmetic Progression (A.P.)

Definition

A sequence (progression), in which every term except the first term differs by the same fixed quantity, called the common difference (c.d.), from its preceding term, is called an "Arithmetic Progression" (simply A.P).

If 'a' is the first term and 'd' is the common difference (fixed quantity), then the general form of A.P. is $a, a + d, a + 2d, ...$

we write
$$T_1 = a$$
$$T_2 = a + d = a + (2 - 1)d$$
$$T_3 = a + 2d = a + (3 - 1)d$$
$$\vdots$$
$$T_n = a + (n - 1)d.$$

T_n is called the n^{th} term with 'a' as the first term and 'd' as the common difference.

- If S_n denotes the sum of the first n terms of the sequence 1, 2, 3, ..., then $S_n = \frac{n(n+1)}{2}$.

- If S_n denotes the sum of the first n terms of the A.P., then, we have

$$S_n = \frac{n}{2}[2a+(n-1)d]$$
$$= \frac{n}{2}[a+a+(n-1)d]$$
$$= \frac{n}{2}\left[T_1+T_n\right].$$

3.2.2 Properties of A.P.

1. If a constant 'k' is added to each term of A.P., with common difference 'd', then the resulting sequence also will be in A.P., with common difference $(d + k)$.

2. If every term is multiplied by a constant 'k', then the resulting sequence will also be in A.P., with the first term 'ka' and common difference 'kd'.

3.2.3 Arithmetic Mean (A.M.)

Definition

If $a_1, a_2, a_3, \ldots, a_n$ are n real numbers, then $\frac{a_1+a_2+a_3+....+a_n}{n}$ is called the arithmetic mean of $a_1, a_2, a_3, \ldots, a_n$.

We observe that the arithmetic mean of three consecutive terms of an A.P., is the middle term. In other words, if a, b, c are three consecutive terms of an A.P., then b is the A.M. of a, b, c and $b = \frac{c+a}{2}$.

Write $b = a + d$ and $c = a + 2d$.

Then A.M. $= \frac{a+b+c}{3} = \frac{a+(a+d)+(a+2d)}{3} = a+d.$

Therefore, if a, c are real numbers, then $b = \frac{c+a}{2}$ is the A.M. of a and c and a, b, c are in A.P.

- Let a, b be any two real numbers and n be a positive integer.

  Suppose, there exist n numbers $a_1, a_2, a_3, \ldots, a_n$ such that $a, a_1, a_2, a_3, \ldots, a_n, b$ are in A.P. Then $a_1, a_2, a_3, \ldots, a_n$ are called **n arithmetic means** between a and b.

  Given a, b and n, let $d = \frac{b-a}{n+1}$.

  Write $a_k = a + kd$, where $k = 1, 2, 3, \ldots, n$.

Then $a, a_1, a_2, a_3, \ldots, a_n, b$ are in A.P. with common difference equal to $\frac{b-a}{n+1}$.

Thus, given a, b and n, there always exist n arithmetic means between a and b.

We observe that $a_1 + a_2 + a_3 + \ldots + a_n = \frac{n}{2}(a+b)$ and

$$a_k = \frac{(n+1-k)a+kb}{n+1}.$$

The following observations are useful in solving problems.

- 3 consecutive members of A.P. can be written as $a - d$, a, $a + d$.
- 4 consecutive members of A.P. can be written as $a - 3d$, $a - d$, $a + d$, $a + 3d$.
- 5 consecutive members of A.P. can be written as $a - 2d$, $a - d$, a, $a + d$, $a + 2d$.

3.2.4 Geometric Progression (G.P.)

Definition

A sequence in which each term except the first term bears a constant ratio to its preceding term is called a Geometric Progression (G.P.). The constant ratio is called the common ratio.

If 'a' is the first term and 'r' is the common ratio, then the general form of G.P. is a, ar, ar^2, ar^3 .....

$T_1 = a = a \cdot r^{1-1}$

$T_2 = ar = a \cdot r^{2-1}$

$T_3 = ar^2 = a \cdot r^{3-1}$

$\vdots$

$T_n = a \cdot r^{n-1}$ is the general term of G.P.

- The sum to n terms of a G.P. is denoted as S_n and $S_n = \frac{a(r^n-1)}{r-1}$, if $r \neq 1$.
- If $r = 1$, then $S_n = a + a + a + \ldots$ (n terms) $= na$
- If $|r| < 1$, the sum to ∞, S_∞ of the infinite geometric series $a, ar, ar^2, \ldots$, is given by $S_\infty = \frac{a}{1-r}$.

3.2.5 Geometric Mean (G.M.)

Definition

If a and b are any two positive numbers, then $\sqrt{ab}$ is the Geometric Mean (G.M.) of a and b, since $a, \sqrt{ab}, b$ are in G.P.

- If $a, g_1, g_2, g_3, ..., g_n, b$ are in G.P., then $g_1, g_2, g_3, ..., g_n$ are called n geometric means between a and b.

  Given a, b and n, let $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$.

  Write $g_k = a \cdot r^k$, where $k = 1, 2, 3, ..., n$, then $a, g_1, g_2, g_3, ..., g_n, b$ are in G.P. with common ratio equal to $\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$.

  Thus, given a, b and n, there always exist n geometric means between a and b.

  Also, it can be observed that $g_1 g_2 g_3 \cdots g_n = (ab)^{n/2}$.

3.2.6 Properties

1. If each term of a G.P. is multiplied (or divided) by a non-zero constant k, then the resulting sequence forms a G.P. with the same common ratio as the initial G.P. This implies that if $a, ar, ar^2, ...$ are in G.P. then $ka, kar, kar^2, ...$ also are in G.P.
2. The reciprocals of the terms of a G.P. also are in G.P.
3. If each term of a G.P. is raised to the power 'k', then the resulting sequence is in G.P. with common ratio r^k.

As in the case of A.P., the following observations are useful.

4. 3 consecutive members of a G.P. : $\frac{a}{r}, a, ar$.
5. 4 consecutive members of a G.P. : $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.
6. 5 consecutive members of a G.P. : $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$.

3.2.7 Relationship between AM and GM

Let A and G denote the AM and GM respectively between two given positive real numbers 'a' and 'b'. Then we have $\text{A} = \frac{a+b}{2}$ and $\text{G} = \sqrt{ab}$. Let us consider

$$A - G = \frac{a+b}{2} - \sqrt{ab}$$

$$= \frac{a+b-2\sqrt{ab}}{2}$$

$$= \frac{(\sqrt{a}-\sqrt{b})^2}{2}$$

$$\geq 0. \qquad \text{.... (1)}$$

This implies that $A \geq G$.

Therefore A.M. of two positive numbers is $\geq$ their G.M.

From (1) we observe that $A = G \Leftrightarrow A - G = 0 \Leftrightarrow \frac{(\sqrt{a}-\sqrt{b})^2}{2} = 0 \Leftrightarrow a = b$.

More generally, let $a_1, a_2, ..., a_n$ be n positive numbers. Let $A = \frac{a_1 + ... + a_n}{n}$ = arithmetic mean and $G = (a_1\, a_2 \,....\, a_n)^{1/n}$ = geometric mean of $a_1, a_2, ..., a_n$. Then it can be shown that $A \geq G$ with equality iff $a_1 = a_2 = ... = a_n$.

3.2.8 Sum to *n* terms of some standard series

Notation : In the begining we mentioned that for any n numbers $a_1, a_2, ..., a_n$.

$a_1 + a_2 + ... + a_n = \sum_{k=1}^{n} a_k$. At times we denote the sum $a_1 + a_2 + ... + a_n$ by Σa_n there by implying that the sum is considered for the terms $a_1, a_2, ... , a_n$.

Thus when n is understood to be given and

$$a_k = 1, \quad \Sigma 1 = \sum_{k=1}^{n} 1 = 1 + 1 + ... + 1 \ (n \text{ terms}) = n.$$

$$a_k = k, \quad \Sigma n = \sum_{k=1}^{n} k = 1 + 2 + ... + n = \frac{n(n+1)}{2}.$$

$$a_k = k^2, \ \Sigma n^2 = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + ...+ n^2 = \frac{n(n+1)(2n+1)}{6}.$$

$$a_k = k^3, \ \Sigma n^3 = \sum_{k=1}^{n} k^3 = 1^3 + 2^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Chapter 4

Mathematical Reasoning

Introduction

Logic is the subject (discipline) that deals with the methods of reasoning and it provides the rules for determining whether an argument made in favour of truth/falsehood of a certain statement is valid or not. Logic is the basis of mathematical reasoning.

We communicate our ideas or thoughts in one or more than one sentence. These sentences are as follows.

(i) Declarative sentence

A sentence that makes a declaration is called a *declarative sentence.*

1. It is a boundary.
2. Hyderabad is the capital of Andhra Pradesh state.
3. $1^2 + 2^2 + 3^2 + 4^2 + ... + n^2 = n(n + 1)/2$.

are some examples of declarative sentences.

(ii) Imperative sentence

A sentence that expresses a command or request is called an *imperative sentence.*

1. Close the door.
2. Stop talking.
3. Please give me your pen.

are some examples of imperative sentences.

(iii) Interrogative sentence (question type)

A sentence in the form of a question is called an *interrogative sentence*.

1. Are you honest in your duty?
2. Where are you going?
3. What are you doing?

are some examples of interrogative sentences.

(iv) Exclamatory sentence

A sentence used to say something loudly and suddenly because of surprise is called an *exclamatory sentence*.

1. How wonderful it is!
2. How beautiful it is!
3. How dangerous it is!

are some examples of exclamatory sentences.

4.1 Definition (Statement)

A declarative sentence is called a statement if it is either true or false but not both.

1. Example: "The sum of two natural numbers is a natural number" is a statement because it is a declarative sentence and it is true.

2. Example: "$5 + 6 > 13$" is also a statement because it is a declarative sentence and the sentence is false.

4.1.1 Note

A statement may be pertaining to mathematics such as $1 + 2 = 3$ or non-mathematical themes. For example, "The sun rises in the east".

4.1.2 Note

A statement is also called as a proposition.

4.1.3 Definition (Truth value)

The truth and falsity of a statement is called its truth value.

4.1.4 Notation

(i) If a statement is true then its truth value is denoted by T, otherwise its truth value is denoted by F.

(ii) The statements are denoted by $p, q, r, s, \ldots$..

For example, p : the sum of 2 and 3 is 5.

4.1.5 Example

Consider the following sentences.

I. **1.** The sum of 1 and 2 is 3.

2. The square of an odd integer is odd.

3. A ball thrown in the open ground will fall on to it.

II. **4.** 16 can be written as the sum of two even prime numbers.

5. $3 = 4$.

6. The sum of all angles of a triangle is 360^0 .

III. **7.** n is a prime.

8. $x + 1 = 6$.

9. $x + y = 8$.

IV. **10.** How long this river is!

11. Tomorrow is Friday.

12. Man will reach Mars by 2020.

13. e is a special number.

Here (1),(2),(3) are statements whose truth value is T and (4),(5),(6) are also statements but their truth value is F. Now (10) is not a statement because it is not a declarative sentence. The declarative sentence (11) has truth value T when it is considered on Thursday otherwise its truth value is F. Therefore (11) is not a statement. The sentence (12) is a statement. The truth value of (12) could be determined only in the year 2020 or earlier if a man reaches Mars before that date. The sentence (13) is not a statement as the word "special" is undefined.

The sentences (7), (8) and (9) are not statements. However, they become statements once a numerical value is assigned to n, x and, x and y respectively. For example if $n = 6$ then (7) is false and if $n = 11$ then (7) is true. The sentence (8) is true for only $x = 5$ and for all other values of x it is false. The sentence (9) has truth value T if $x = 5$, $y = 3$ and has truth value F if $x = 1$, $y = 3$.

The sentences of the type (7), (8) and (9) are called predicates which we will discuss later. The symbols which need to be given values from a given set (it is known as Universe) in order to obtain a statement are called free variables. The predicates in one free variable, two free variables, three free variables respectively are denoted in the form $P(x)$, $P(x, y)$, $P(x, y, z)$. For example, $P(n)$: n is prime, $n \in \mathbf{N}$ (set of natural numbers is the universe), $Q(x)$: $x + 1 = 6$, $x \in \mathbf{Z}$ (set of integers is the universe) $S(x, y) : x + y = 6$, $x, y \in \mathbf{R}$ (set of real numbers is the universe).

4.1.6 The law of the excluded middle, the law of contradiction

In mathematical reasoning, we are not going to be preoccupied with the actual truth value of a statement. We shall be interested only in the fact that it has a truth value. Therefore, we ignore the sentences like (10) to (13). Mathematical theories are constructed starting with some fundamental assumptions called axioms.

For example, the natural numbers are generated by Peano's axioms in number theory and later the whole numbers, integers, rational, irrational and real numbers are defined. Further addition, subtraction etc.. are defined. Likewise, for definiteness we have the following two assumptions for the statements (or propositions) .

1. **For every statement (or proposition) *p*, either *p* is true or** its negation (See 4.2.1) is true; There is no third possibility. This is known as the **law of the excluded middle**.
2. **For every statement (or proposition) *p*,** that *p* is true and *p* is not true are mutually exclusive. It is known as the **law of contradiction**.

With these assumptions, sentences of ambiguous nature are eliminated from our discussion. The theory of mathematical reasoning is developed by defining certain terms involving the statements. This we learn in the next section.

4.2 Negation, Conjunction and Disjunction

In case of number theory, after defining the real numbers we go to operators like + (addition), – (subtraction), × (multiplication) etc.,. Likewise, we have logical operators or connectives for combining or modifying the statements. In this section we learn ***negation, conjunction*** and ***disjunction*** **and learn *if ... then, if and only if*** in the next section. The negation modifies a statement and others combine the statements. The statements are combined by means of ***and, or, if.. then, and if and only if***. The statements are modified by the word *not*. Now we proceed to give the definitions of these.

4.2.1 Definition (Negation)

The denial of a statement p is called the negation of p and it is denoted by ~p read as not p.

Example : Let *p* : Mumbai is a city. Then ~*p*: Mumbai is not a city.

Note that ~*p* can also be stated as "It is false that Mumbai is a city" or "It is not the case that Mumbai is a city"

4.2.2 Definition (Simple statement)

A statement is said to be a simple statement if it cannot be split into two or more sentences.

For example, "Einstein is a genius" is a simple statement as it cannot be split into two or more sentences.

While expressing various ideas, we use two (or more) statements one for each idea and combine them by connecting words like 'and', 'or' etc.

1. Example: Let p : I had reached Mumbai and travelled by train to reach my house. This statement p includes two simple statements.

q : I had reached Mumbai.

r : I travelled by train to reach my house.

2. Example: Let p : all primes are either even or odd.

This statement p includes two simple statements.

q : All prime numbers are odd

r : All prime numbers are even.

4.2.3 Definition (Compound Statement)

*A **compound statement** is a statement which is made up of two or more simple statements which are called components of the given statement.*

The words (phrases) connecting the components are called connectives.

4.2.4 Definition (Propositional function)

*A **propositional function** is a function whose variables are statements such as p, q, r, s, ... for example, if p, q are statements "p and q" is a propositional function of p and q. It is denoted by P(p, q).*

4.2.5 Definition (Truth table)

A table showing all possible truth values of the components of a propositional function P and its truth values is called the truth table.

For example, the truth table of $\sim p$ is as follows.

Truth table of $\sim p$

p	$\sim p$
T	F
F	T

Here $\sim p$ is a propositional function of p. It may be expressed as P(p): $\sim p$.

4.2.6 Definition (Equivalent statements)

*Two statements are said to be **equivalent** if the two statements have the same truth values. In case the statements are propositional functions, they are equivalent if they have the same truth table. If two propositional functions P and Q are equivalent we denote it as $P \equiv Q$.*

For example p and $\sim(\sim p)$ are equivalent statements i.e., $p \equiv \sim(\sim p)$. It can be verified by construction of the truth table.

Truth table of $\sim(\sim p)$

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

Note that the truth values of p and $\sim(\sim p)$ are the same.

4.2.7 Definition (Conjunction)

If p, q are statements then p and q is a statement and it is called as conjunction of p and q. It is denoted by $p \wedge q$.

We read $p \wedge q$ as p and q. The conjunction of p and q has truth value T only when both p and q have truth values T. The truth table of $p \wedge q$ is shown in the following table.

Truth table of $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

$p \wedge q \wedge r$ is said to be conjunction of the statements p, q and r and this can be extended to a finite number of statements.

We can consider $p \wedge q$ as a propositional function of p and q i.e., $P(p, q) : p \wedge q$.

4.2.8 Example

Let P : 0 is less than every positive number and every negative number.

Let the components of the statement P be

q : 0 is less than every positive number

r : 0 is less than every negative number.

We know that q is true and r is false. Therefore, the truth value of P is false.

4.2.9 Example

Let P: Through two given points only one straight line can be drawn and through three non-collinear points only one circle can be drawn.

Let the components of P be

q : Through two given points only one straight line can be drawn

r : Through three non-collinear points only one circle can be drawn.

The statements q and r are true, therefore the truth value of P is T.

4.2.10 Example

Let p : Mixture of spirit and water can be separated by chemical methods.

Here, p is not a compound statement. The statement p does not have two statements. The word "and" used in p, is not a connective. It tells only about the contents in the single word mixture. Note the difference between the literal use of the word and in contrast with its usage as conjunction.

4.2.11 Definition (Disjunction)

If p and q are any two statements then p or q is defined as disjunction of p and q and is denoted by $p \vee q$. $p \vee q$ has truth value T whenever atleast one of p, q has truth value T. The truth table of $p \vee q$ is shown in the following table.

Truth table of $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

$p \vee q \vee r$ is the disjunction of the statements p, q and r. Note that, the disjunction is defined in the inclusive sense i.e., disjunction connective is inclusion or.

There are two types of 'or' which we use in our every day life.

4.2.12 Example

Let p : A cup of coffee or tea is available with snacks in a restaurant. From the above statement we understand that there is a choice between coffee and tea. One can have coffee with snacks or tea with snacks but can not have both coffee and tea. 'or' used in such contexts, is called exclusive 'or'.

4.2.13 Example

Let p : A student who has taken Biology or Chemistry at degree level can apply for P.G. course in Microbiology.

We understand from this statement that a student who has taken either Biology and / or Chemistry can apply for the said course. So, a student who has taken both the subjects can apply for the said course. Or used in such contexts is called ***inclusive 'or'***.

4.2.14 Example

Let p : 50 is a multiple of 7 or 8. Let its components be

q : 50 is a multiple of 7

r : 50 is a multiple of 8 .

Then $p \equiv q \vee r$. We know that both q and r are false. Therefore p is false.

4.2.15 Example

Let p : Two distinct points in space determine a line or a plane.

Let its component be

q : two points in space determine a line.

r : two points in space determine a plane.

Then $p \equiv q \vee r$. We know that q is true but r is false. Since either of q and r is true the truth value of p is T.

4.2.16 Example

Let p : The college is closed if it is a holiday or a Sunday.

Let its components be

q : The college is closed if it is a holiday.

r : The college is closed if it is a Sunday.

The truth value of p is T since both q and r are true.

4.3 Implication - conditional and bi-conditional

We come across, in everyday life, sentences like "**if** I had gone to the railway station in time **then** I would not have missed the train", "**if** x is a real number **then** $x^2 \geq 0$ and "**If** a number is divisible by 49 **then** it is divisible by 7.

4.3.1 Definition : Implication

Let p and q be any two statements. "If p then q" is called an implication and it is denoted by $p \to q$.

We define $p \to q$ as a statement and it is false only when p is true and q is false.

The truth table of $p \to q$ is shown in the following table.

Truth table of $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

4.3.2 Note

1. Implication is also called conditional.
2. For the implication $p \to q$, p is called premise or hypothesis or an antecedent and q is called as conclusion or consequent.
3. The truth value of the implication is defined. Therefore, the truth value of an implication depends only on the truth values of p and q but not any relation between antecedent and consequent of the implication. For example, if Andhra Pradesh is in America then 5 + 6 = 17. The truth value of this implication is T. The reason is, the statement "Andhra Pradesh is in America" has truth value F and the statement "5 + 6 = 17" has the truth value F.

   Of course, there are rules for testing the validity of the conclusions which you learn in higher classes.
4. In our everyday language, it is customary to have some sort of relation between antecedent and consequent of an implication. For example, "if I get a ticket then I shall see the movie". In this implication, the consequent (see the movie) refers to the antecedent (get a ticket). But this is not

the case in our defined implication. i.e., The statements p and q of an implication $p \to q$ **need not** have any relationship.

Therefore, there may or may not be a relation between the antecedent and consequent as per the definition of an implication. For example, if Einstein is a genius then 5 + 6 = 3. This statement does not make sense to us in our conventional language. However, according to our definition of implication (conditional), this statement is considered as an implication.

5. The statements, "if p then q" and "p implies q" are not the same in the reasoning. But, in Mathematics they are used interchangeably.

1. Example : If ABC is a triangle then sum of its angles is 180^0.

2. Example : If the sky is overcast then the Sun is not visible.

4.3.3 Definitions

Let p and q be any two statements

(i) $q \to p$ is called the *converse* of $p \to q$.

(ii) $\sim p \to \sim q$ is called *opposite* of $p \to q$.

(iii) $\sim q \to \sim p$ is called *contrapositive* of $p \to q$.

3. Example: Consider the implication "if a number is divisible by 36 then it is divisible by 6". Let

p : A number is divisible by 36.

q : A number is divisible by 6.

Then the given implication is $p \to q$. The converse statement of it is $q \to p$. i.e., if a number is divisible by 6 then it is divisible by 36.

4. Example : If $x = a$ is a root of $f(x) = 0$ then $(x - a)$ is a factor of $f(x)$.

The opposite of it is if $x = a$ is not a root of $f(x) = 0$ then $(x - a)$ is not a factor of $f(x)$.

5. Example: Consider the implication "if a number is divisible by 25 then it is divisible by 5". Let

p : A number is divisible by 25.

q : A number is divisible by 5.

Then the given implication is $p \to q$. Its contrapositive statement is $\sim q \to \sim p$ i.e., "if a number is not divisible by 5 then it is not divisible by 25".

4.3.4 Definition (Biconditional)

Let p and q be any two statements. The biconditional is defined as a conjunction of $p \to q$ and $q \to p$ and it is denoted by $p \leftrightarrow q$. i.e.,

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$. Further, $p \leftrightarrow q$ is also stated as "p if and only if q" or "p is a necessary and sufficient condition for q" and vice-versa. The truth table of $p \leftrightarrow q$ is given below.

Truth table of $p \leftrightarrow q$

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note that $p \leftrightarrow q$ is true only when p and q have the same truth values.

6. Example: Let p: A number (three or more digits) in which the number formated by the digits in the last two places (unit place and tens place) is divisible by 4 and q : the number is divisible by 4. Then $p \to q$ means if a number in which the part formed by the last two digits is divisible by 4, then the number is divisible by 4 and ($q \to p$) means if the number is divisible by 4 then the number in the last two places of the given number is divisible by 4. The conjunction of these two statements may be stated as "a number is divisible by 4 if and only if the number formed by the digits in the last two places is divisible by 4". In symbols $p \leftrightarrow q$.

4.4 Quantifiers

The phrases like "there exists" denoted by $\exists$, and "for all" denoted by $\forall$ are called quantifiers. The symbol $\exists$ is also called *existential quantifier* and $\forall$ is called *Universal quantifier*. $\forall$ is also used in lieu of the phrase "for every".

4.4.1 Definition: Open statement

Let U be a set. Suppose a predicate p(x) is true or false for each $x \in U$. Such p(x) is called as an open statement on U. Then the set U is called the Universe of discourse or the Universe of the open statement p.

For example, $p(x)$: x is a prime and the universe U is the set of all positive integers i.e., A declarative sentence is an open statement if (i) it contains one or more variables (ii) when the variables are assigned values from a set (it is the universe) then it has truth value T or F.

The following are examples of open statements:

1. $p(x) : x + 2$ is an even number, where the universe is the set all natural numbers.
2. $p\ (x, y) : x < y$, where x and y are members of the set of all real numbers.

4.4.2 Notation

(i) "$\forall x, p(x)$" is used in the sense that $p(x)$ is true for every x belonging to the universe of p.

(ii) "$\exists x, p(x)$" is used in the sense that $p(x)$ is true for atleast one x belonging to the universe of p.

4.4.3 Note

Negation of quantified statements

1. $\sim [\forall x,\ p(x)] \ \equiv\ \exists x,\ \sim p(x)$
2. $\sim [\exists x,\ p(x)] \ \equiv\ \forall x,\ \sim p(x)$
3. $\sim [\forall x,\ \sim p(x)] \ \equiv\ \exists x,\ p(x)$
4. $\sim [\exists x,\ \sim p(x)] \ \equiv\ \forall x,\ p(x)$

4.5 Validating Statements

Methods of proof of an implication $p \rightarrow q$

I. Trivial proof of $p \rightarrow q$.

In this method of trivial proof of $p \rightarrow q$, it is enough to show that the truth value of q is true i.e., the implication $p \rightarrow q$ is true when q is true (regardless of the truth value of p).

1. **Example:** Let $p(a, b) : a, b$ are non-zero integers such that $a \geq b$.

   $q(a, b) : a^0 \geq b^0$ and the universe U is the set of all non-zero integers.

   Then q is true. Therefore, the implication $p(a, b) \rightarrow q(a, b)$ is true (by trivial method of proof).

II. Vacuous proof of $p \rightarrow q$.

In this method p is shown to be false so that the implication $p \rightarrow q$ is true.

2. **Example:** To show that $\phi \subseteq$ A, we have to show that

   If $\forall\ x,\ x \in \phi$ then $x \in$ A .... (1)

   Let $p(x) : x \in \phi$ and $q(x) : x \in$ A.

Then we have to show $\forall x,\ \ p(x) \Rightarrow q(x)$. $p(x)$ is false, since the null set has no elements in it. Therefore (1) is true by the method of vacuous proof. Hence, $\phi \subseteq$ A.

III. Direct Method of proof of an implication $p \rightarrow q$

In this method, assuming p is true, q is shown to be true so that $p \rightarrow q$ is true.

3. Example: Suppose we have to show that the implication "if n is odd then n^2 is odd" is true for any integer n using the direct method. Let $p(n)$: n is odd and $q(n)$: n^2 is odd. Here the universe U is the set of all integers. Assume that p is true. Then n is odd. Therefore there exists some integer k such that $n = 2k + 1$.

Now, $n^2 = (2k + 1)^2$

$= 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2m + 1,$

where $m = 2k^2 + 2k$. Thus $n^2 = 2m+1$.

Hence n^2 is odd.

Therefore, the given implication is true.

IV. Contrapositive (Indirect) of method of) proof of an implication $p \rightarrow q$.

In this method assuming that q is false, the conclusion p is shown to be false.

The logic involved in it is, the implication $p \rightarrow q$ and $\sim q \rightarrow \sim p$ are equivalent i.e., $p \rightarrow q \equiv \sim q \rightarrow \sim p$. It is called **contrapositive law**.

It can be shown by constructing the truth table of these two implications which are given below.

Let P(p, q) : $p \rightarrow q$ and Q(p, q): $\sim q \rightarrow \sim p$.

p	q	$\sim p$	$\sim q$	P : $p \rightarrow q$	Q: $\sim q \rightarrow \sim p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The truth values of P(p, q) and Q(p, q) are the same. Therefore they are equivalent.

4. Example: Suppose we have to show that the implication "if $(3n + 2)$ is odd then n is odd" where n is any integer, by contrapositive method. Let $p(n)$: $(3n + 2)$ is odd, $q(n)$: n is odd and the universe U is the set of all integers. Suppose that $q(n)$ is false. Then n is not odd. It means n is an even number. (As an integer n is either even or odd). Since n is even there exists an integer k such that $n = 2k$. $3n + 2 = 3(2k) + 2 = 2(3k + 1)$. Therefore $3n + 2$ is an even number. Hence $p(n)$ is false. By contrapositive law, the given implication is true.

V. Contradiction method of proof of an implication $p \to q$

In this method we assume that p is true and q is false and arrive at a contradiction.

This argument leads to $p \wedge (\sim q)$ is false. Hence $\sim(p \wedge (\sim q))$ is true i.e., $(\sim p) \vee q$ is true. But we have $(\sim p) \vee q \equiv p \to q$. Hence the implication $p \to q$ is true.

In case of a simple statement p, to show that p is true by the method of contradiction we will assume that p is false and arrive a contradiction to a fact or to the contents of the statement p. This leads to conclude that our assumption is wrong. Hence the given statement is true.

VI. Method of proof by counter example

To disprove the statement: $\forall x, p(x)$, it is enough to provide a counter example i.e., to show that $\forall x, p(x)$ is false it is sufficient to exhibit a specific value v in the universe such that $p(v)$ is false. The value v is called counter example to the assertion $\forall x, p(x)$.

The argument is as follows:

$\forall x, \ p(x) \equiv p(a) \wedge p(b) \wedge p(c) \wedge p(d)$ ..... (a, b, c etc. are in the Universe) $\forall x, p(x)$ is true if and only if $p(a), p(b), p(c)$, ... are all true. If any one of $p(a), p(b), p(c)$, ... is false then $\forall x, p(x)$ is false.

So, to disprove $\forall x, p(x)$ we can use the method of proof by a counter example. For example to disprove the statement "all primes are odd numbers" we can find a counter example "2 (which is prime but not odd)". Here the universe U is set of all positive integers.

4.5.1 Note

To prove $p \leftrightarrow q$ is true, we have to prove $p \to q$ and $q \to p$ are true. We may choose any method of proof given above to prove $p \to q$ or $q \to p$.

4.5.2 Note

If we show that $p(x)$ is true for any arbitrary element of the universe U then $p(x)$ is true for all $x \in$ U. Similarly, if we show that $p(x, y)$ is true for any arbitrary elements of x and $y \in$ U then $p(x, y)$ is true for all $x, y \in$ U.

Reference Books

- Mathematics, Text Book for Class XI; National Council of Educational Research and Training (NCERT); New Delhi; 2006.
- Mathematics, Text Book for Class XII, Part - I; National Council of Educational Research and Training (NCERT); New Delhi; 2006.
- Mathematics, Text Book for Class XII, Part - II; National Council of Educational Research and Training (NCERT); New Delhi; 2007.
- Intermediate First Year; Mathematics, Paper I - A, Telugu Akademi; Hyderabad; 2008.
- Matrices - Schaum's Outline Series; Frank Ayres; Mc. Graw Hill Education (India) Ltd., 2007.
- College Algebra - Schaum's Outline series;Murray R. Spiegel and Robert E Moyer; McGraw - Hill Education (India) Ltd.; 2007.
- Vector Analysis -Schaum's outline series; Murrary R. Spiegel; McGraw - Hill Education (India) Ltd.; 2007.
- Elementary Vector Analysis; C.E. Weatherburn; G. Bell & Sons Ltd., London; 1966.
- Trigonometry - Schaum's outline series; Robert E. Moyer; McGraw -Hill Education (India) Ltd.; 2007.
- Plane Trigonometry (Metric edition); S.L. Loney; SBD Publishers & Distributors, New Delhi; 2001.
- Elementary Trigonometry; H.S. Hall & S.R. Knight; Macmillan & Co., London; 1962.

BOARD OF INTERMEDIATE EDUCATION
Syllabus in Mathematics Paper - IA
To be effective from the academic year 2012-13

Name of Topic and Sub Topics	No. of Periods
ALGEBRA	
1. Functions	16
1.1 Types of functions - Definitions	
1.2 Inverse functions and Theorems	
1.3 Domain, Range, Inverse of real valued functions	
2. Mathematical Induction	08
2.1 Principle of Mathematical Induction & Theorems	
2.2 Applications of Mathematical Induction	
2.3 Problems on divisibility	
3. Matrices	28
3.1 Types of matrices	
3.2 Scalar multiple of a matrix and multiplication of matrices	
3.3 Transpose of a matrix	
3.4 Determinants	
3.5 Adjoint and Inverse of a matrix	
3.6 Consistency and inconsistency of Equations - Rank of a matrix	
3.7 Solution of simultaneous linear equations	
VECTOR ALGEBRA	
4. Addition of Vectors	18
4.1 Vectors as a triad of real numbers	
4.2 Classification of vectors	
4.3 Addition of vectors	

4.4 Scalar multiplication	
4.5 Angle between two non zero vectors	
4.6 Linear combination of vectors	
4.7 Component of a vector in three dimensions	
4.8 Vector equations of line and plane including their Cartesian equivalent forms	
5. Product of Vectors	**28**
5.1 Scalar product - Geometrical Interpretations - Orthogonal projections	
5.2 Properities of dot product	
5.3 Expression of dot product in $i,\ j, k$ system - Angle between two vectors	
5.4 Geometrical Vector methods	
5.5 Vector equations of plane in normal form.	
5.6 Angle between two planes	
5.7 Vector product of two vectors and properties	
5.8 Vector product in $i,\ j, k$ system.	
5.9 Vector Areas	
5.10 Scalar Triple product	
5.11 Vector equations of plane in different forms, skew lines, shortest distance and their Cartesian equivalents. Plane through the line of intersection of two planes, condition for coplanarity of two lines, perpendicular distance of a point from a plane, Angle between line and a plane. Cartesian equivalents of all these results.	
5.12 Vector Triple product - Results.	
TRIGONOMETRY	
6. Trigonometric Ratios up to Transformations	**20**
6.1 Graphs and Periodicity of Trigonometric functions	
6.2 Trigonometric ratios and Compound angles	

6.3 Trigonometric ratios of multiple and sub multiple angles	
6.4 Transformations - Sum and Product rules	
7. Trigonometric Equations	**05**
7.1 General solution of Trigonometric Equations	
7.2 Simple Trigonometric Equations - Solutions	
8. Inverse Triogonometric Functions	**07**
8.1 To reduce a Trigonometric function into a bijection	
8.2 Graphs of Inverse Trigonometric Functions	
8.3 Properities of Inverse Trigonometric Functions	
9. Hyperbolic Functions	**04**
9.1 Definition of Hyperbolic Function - Graphs	
9.2 Definition of inverse Hyperbolic Functions - Graphs	
9.3 Addition formulas of Hyperbolic Functions	
10. Properities of Triangles	**16**
10.1 Relation between sides and angles of a Triangle	
10.2 Sine, Cosine, Tangent and Projection rules	
10.3 Half angle formulae and areas of a triangle	
10.4 In-circle and Ex-circle of a Triangle	
Total	**150**

Additional Reading Material for Mathematics - IA

The following topics are there in the Common Core Syllabus which is covered in previous curriculum (IX and X classes) and which is not included in A.P. Intermediate syllabus. For the benefit of students writing competitive examination based on Common Core Syllabus, we are giving the material briefly on the following topics. **No question is to be set in the IPE, Mathematics - IA from this syllabus.**

1. Sets

Introduction

Sets and their representations

The empty set

Finite and infinite sets

Equal sets

Sub sets

Power sets

Universal sets

Venn Diagram

Operations on sets

Complement of a set

Practical problems on union and intersection of two sets

2. Relations

Introduction

Cartesian Product of sets

Relations

3. Sequences and Series

Introduction

Sequences

Series

Arithmetic Progressions (A.P.)

Geometric Progressions (G.P.)

Relationship between A.M. and G.M.

Sum and upto *n* terms of special series

4. Mathematical Reasoning

Introduction

Statement

New statements from old

Special words / phrases

Implications

Validating Statements

BOARD OF INTERMEDIATE EDUCATION A.P. : HYDERABAD
MODEL QUESTION PAPER w.e.f. 2012-13
MATHEMATICS - IA

(English Version)

Time : 3 Hours **Max. Marks : 75**

Note : The Question Paper consists of three sections A, B and C

Section - A $10 \times 2 = 20$ Marks

I. Very Short Answer Questions

(i) Answer **All** questions

(ii) Each Question carries **two** marks

1. If $A = \left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\right\}$ and $f : A \to B$ is a surjection defined by $f(x) = \cos x$ then find B.

2. Find the domain of the real-valued function $f(x) = \frac{1}{\log(2-x)}$.

3. A certain bookshop has 10 dozen chemistry books, 8 dozen physics books, 10 dozen economics books. Their selling prices are Rs. 80, Rs. 60 and Rs. 40 each respectively. Find the total amount the bookshop will receive by selling all the books, using matrix algebra.

4. If $A = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix}$, then find $A + A'$ and $A A'$.

5. Show that the points whose position vectors are $-2\bar{a} + 3\bar{b} + 5\bar{c}$, $\bar{a} + 2\bar{b} + 3\bar{c}$, $7\bar{a} - \bar{c}$ are collinear when $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar vectors.

6. Let $\bar{a} = 2\bar{i} + 4\bar{j} - 5\bar{k}$, $\bar{b} = \bar{i} + \bar{j} + \bar{k}$ and $\bar{c} = \bar{j} + 2\bar{k}$. Find unit vector in the opposite direction of $\bar{a} + \bar{b} + \bar{c}$.

7. If $\bar{a} = \bar{i} + 2\bar{j} - 3\bar{k}$ and $\bar{b} = 3\bar{i} - 2\bar{j} + 2\bar{k}$ then show that $\bar{a} + \bar{b}$ and $\bar{a} - \bar{b}$ are perpendicular to each other.

8. Prove that $\frac{\cos 9^0 + \sin 9^0}{\cos 9^0 - \sin 9^0} = \cot 36^0$.

9. Find the period of the function defined by $f(x) = \tan(x + 4x + 9x + ... + n^2x)$.

10. If $\sinh x = 3$ then show that $x = \log_e(3+\sqrt{10})$.

Section - B

5 × 4 = 20 Marks

II. Short Answer Questions

(i) Answer any Five questions.

(ii) Each Question carries Four marks.

11. Show that $\begin{vmatrix} bc & b+c & 1 \\ ca & c+a & 1 \\ ab & a+b & 1 \end{vmatrix} = (a-b)(b-c)(c-a)$.

12. Let A B C D E F be regular hexagon with centre 'O'. Show that $\overline{AB}+\overline{AC}+\overline{AD}+\overline{AE}+\overline{AF} = 3\overline{AD} = 6\overline{AO}$.

13. If $\bar{a} = \bar{i} - 2\bar{j} - 3\bar{k}, \bar{b} = 2\bar{i} + \bar{j} - \bar{k}$ and $\bar{c} = \bar{i} + 3\bar{j} - 2\bar{k}$ find $\bar{a} \times (\bar{b} \times \bar{c})$.

14. If A is not an integral multiple of $\frac{\pi}{2}$, prove that

(i) $\tan A + \cot A = 2\,\text{cosec}\, 2A$

(ii) $\cot A - \tan A = 2\cot 2A$

15. Solve : $2\cos^2\theta - \sqrt{3}\sin\theta + 1 = 0$.

16. Prove that $\cos\left(2\tan^{-1}\frac{1}{7}\right) = \sin\left(4\tan^{-1}\frac{1}{3}\right)$.

17. In a ΔABC prove that $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}$.

Section - C

5 × 7 = 35 Marks

III. Long Answer Questions

(i) Answer any Five questions.

(ii) Each Question carries Seven marks.

18. Let $f : A \to B$, $g : B \to C$ be bijections. Then prove that $(gof)^{-1} = f^{-1}og^{-1}$.

19. By using mathematical induction show that $\forall n \in N$, $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+....$ upto n terms $= \frac{n}{3n+1}$.

20. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$ then find $(A')^{-1}$.

21. Solve the following equations by Gauss-Jordan method $3x + 4y + 5z = 18$, $2x - y + 8z = 13$ and $5x - 2y + 7z = 20$.

22. If A = (1, –2, –1), B = (4, 0, –3), C = (1, 2, –1) and D = (2, –4, –5), find the distance between $\overline{AB}$ and $\overline{CD}$.

23. If A, B, C are angles of a triangle, then prove that $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2} = 1 - 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$.

24. In a ΔABC, if $a = 13$, $b = 14$, $c = 15$, find R, r, r_1, r_2 and r_3.

1.11 Some Problems on Set Theory

1.11.1 Show that the set of letters needed to spell "CATARACT" and the set of letters needed to spell "TRACT" are equal.

Solution: Let X be the set of letters in "CATARACT". Then

$$X = \{C, A, T, R\}$$

let Y be the set of letters in "TRACT"

$$Y = \{T, R, A, C\}$$

since every element in X is in Y and every element in Y is in 'X' therefore

$$X = Y$$

1.11.2 Show that $A \cup B = A \cap B$ implies $A = B$

Solution: Let $x \in A \Rightarrow x \in A \cup B \Rightarrow x \in A \cap B$ ($\because A \cup B = A \cap B$)

$\Rightarrow x \in B$

Therefore $A \subset B$ .... (1)

let $x \in B \Rightarrow x \in A \cup B \Rightarrow x \in A \cap B (\because A \cup B = A \cap B) \Rightarrow x \in A$

Therefore $B \subset A$ .... (2)

From (1) and (2) $A = B$

1.11.3 For any sets A and B show that $\rho(A \cap B) = \rho(A) \cap p(B)$ (Here (ρ(A) is power set of A

Solution: Let $X \in \rho(A \cap B) \Rightarrow X \subset A \cap B$ (By the definition of power set)

$\Rightarrow X \subset A$ and $X \subset B$

$\Rightarrow X \in \rho(A)$ and $X \in \rho(B)$

$\Rightarrow X \in \rho(A) \cap \rho(B)$

Therefore $\rho(A \cap B) \subset \rho(A) \cap \rho(B)$ ... (1)

Let $Y \in \rho(A) \cap \rho(B) \Rightarrow Y \in \rho(A)$ and $Y \in \rho(B)$

$\Rightarrow Y \subset A$ and $Y \subset B$

$\Rightarrow Y \in \rho(A \cap B)$

Therefore $\rho(A) \cap \rho(B) \subset \rho(A \cap B)$ ... (2)

From (1) and (2) $\rho(A \cap B) = \rho(A) \cap \rho(B)$

1.11.4 If A is any set such that $n(\rho(A)) = 64$ then find n(A)

Solution: $n(\rho(A)) = 64 = 2^6 = 2^n$

Therefore $n(A) = 6$

1.11.5 Show that if $A \subset B$ then $C - B \subset C - A$

Solution: Let $x \in C - B \Rightarrow x \in C$ and $x \notin B$

$(\because x \notin B \Rightarrow x \notin A(\because A \subset B))$

$\Rightarrow x \in C - A$

Therefore $C - B \subset C - A$

1.11.6 Assume that $\rho(A) = \rho(B)$. Show that A = B

$$A \in \rho(A) \text{ (obvious)}$$

$$\Rightarrow \quad A \in \rho(B) \text{ (since } \rho(A) = \rho(B)\text{)}$$

$$A \subset B \text{ (by definition of power set)}$$

Similarly, by the same reasoning above we can prove that $B \subset A$

Therefore A = B

1.11.7 Let A, B and C be the sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$ then show that B = C

1.11.8 Let A, B and C be the sets such that $\rho(A \cup B) = \rho(A) \cup \rho(B)$. Justify your answer

1.11.9 Show that the following four conditins are equivalent:

(i) $A \subset B$ (ii) $A - B = \phi$ (iii) $A \cup B = B$ (iv) $A \cap B = A$

1.11.10 In a class of 100 students 55 students have passed in Mathematics and 67 students passed in Physics. Then find the number of students who have passed in Physics only.

1.11.11 Out of 800 boys in a school, 224 played cricket, 240 played basketball. Of the total 64 played both basketball and hockey; 80 played cricket and basketball and 40 played cricket and hockey, 24 played all the three games. Find the number of boys who did not play any game.

1.11.12 The set of intelligent students in a class is

(1) a null set (2) a singleton set

(3) a finite set (4) not a well-defined collection

1.11.13 If aN = {ax : x $\in$ N} then

$3N \cap 7N$ = (1) 21 N (2) 10N (3) 4N (4) none

1.11.14 $A = \{4^n - 3n - 1 : n \in N\}$ $B = \{9(n-1) : n \in N\}$ then $A \cup B = ?$

1.11.15 Two finite sets have m and n elements. The total number of subsets of the first set is 56 more then the total number of subsets of the second set. The values of m and n are

1.11.16 If A and B have 3 and 6 elements then the minimum number of elements in $A \cup B$ is

1.11.17 A survey shows that in a city 63% of the citizens like tea where as 76% like coffee. If x% like both tea and coffee, then

1) x = 63 2) x = 39 3) $5050 \le x \le 63$ 4) $39 \le x \le 63$

1.11.18 Suppose $A_1, A_2, \ldots.. A_{30}$ are 30 sets each having five elements and $B_1, B_2, \ldots.B_n$ are n sets each having 3 elements such that $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$. If each elements of S belongs to exactly ten of A_i's and exactly 9 of the B_j's then the value of n is

1.11.19 If A = {ϕ, {ϕ}} then the power set of A is ....

1.11.20 Let $A = \{(x,y) : y = e^x, x \in R\}, B = \{(x,y) : y = e^{-x}, x \in R\}$ then $A \cap B =$

Answers

1.11.10) 45 1.11.11) 160 1.11.12) 4 1.11.13) 1

1.11.14) B 1.11.15) 6,3 1.11.16) 6 1.11.17) 4

1.11.18) 45 1.11.19) {ϕ, {ϕ}, {{ϕ}}, A} 1.11.20) $\neq \phi$

2.3 Solved Problems on Relations

2.3.1 Let A and B be two sets containing 2 elements and 4 elements respectively. Then find the number of subsets of $A \times B$ having 3 or more elements.

Solution: n(A) = 2. n(B) = 4. So $n(A \times B) = 8$

Therefore, number of subsets of

$$A \times B = 2^8 = 256$$

Therefore, number of subsets of $A \times B$ having less than 3 elements

$$= 8_{c_0} + 8_{c_1} + 8_{c_2} = 1 + 8 + 28 = 37$$

Therefore, number of subsets of $A \times B$ having 3 or more elements

$$= 256 - 37 = 219$$

2.3.2 Find the domain and range of the relation $\{(1,x),(2,y),(3,x),(4,z)\}$

Solution: Domain $= \{1,2,3,4\}$

Range $= \{x, y, z\}$

2.3.3 If A = {1, 2, 3} find the number of reflexive relations in A

Solution: The number of reflexive relations ina set A having n elements

$$= x.\log_{10}\frac{10}{5} + \log_{10}(1 + 2^x) = \log_{10} 6$$

Therefore, answer is $2^{3(3-1)} = 2^6 = 64$

2.3.4 Let R be the set of real numbers. Show that

$A = \{(x, y) \in RxR : x = ay, a$ for some rational number$\}$ is not an equivalence relation?

Solution: (i) $xAx \rightarrow x = \alpha x$ where $\alpha = 1 \in Q \rightarrow A$ is reflexive

(ii) $(0,1) \in A$ since $0 = 0.1$. But $(0,1) \notin A$ is not symmetric

$\therefore$ A is not an equivalance relation

2.3.5 $A = \{(x, y) \in RxR : x - y$ is an integer$\}$. Show that A is an equivalence relation

Solution: (i) $xAx \Rightarrow x - x = 0 \in Z \Rightarrow A$ is reflexive

(ii) $xAy \Rightarrow x - y \in Z \Rightarrow y - x \in Z \Rightarrow A$ is symmetric

(iii) $xAy, yAz \Rightarrow x - y \in Z, y - z \in x - y + y - z \in Z \Rightarrow x - z \in Z$

$\Rightarrow A$ is transitive

$\therefore$ A is an equivalance relation on R

2.4 Exercise

2.4.1 R = {(1,3),(4,2),(2,4),(2,3),(3,1)} be a relation on the set A = {1,2,3,4}. The relation R is

(1) A function (2) Reflexive (3) not symmetric (4) transitive

2.4.2 The set S = {1,2,3, ... 12} is to be partitioned in to three sets A, B, C of equal size. Thus $A \cup B \cup C = S, A \cap B = B \cap C = C \cap A = \phi$. The number of ways to partition S is?

2.4.3 Let R be the relation on the set R of real numbers defined by a Rb iff $|a-b| \leq 1$ show that R is reflexive and symmetric.

2.4.4 Show that the relation less than in the set of natural numberes is only transitive

2.4.5 Let N denote the set of all natural numbres and R be the relation on N X N defined by (a,b) R (c,d) if ad(b+c) = bc (a+d) then show that R is an equivalence relation.

Chapter 5 : Logarithms

5.1 Introduction

The rebirth of science after the state of rigor mortis imposed by the church in the Middle ages was particularly evident in the awakened interest in astronomy and in the attendant development of trigonometry, generated also by other forms of world exploration - land surveying, cartography and navigation.

scientists every where began to spend enormous amounts of time in calculating tables of trigonometric functions, and it became important to find methods of replacing the often laborious operations of multiplicatin and division with addition and subtraction - e.g, by employing formulae such as $2 \sin\alpha \sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$.

5.2 History of Logarithms

The invention of Logarithms is widely attributed to at least two people viz., John Napier (1550 - 1617) and Joost Burgi (1552-1632). John Napier originally published his work on logarithms to calculate sines of right angled triangles that have very large hypotenuses. Later, Napier along with another mathematician, Henry Brigggs modified the definiton of logarithms intot the form that we know today.

5.3 Definition

Logarithm of a positive real number u to a positive base $x \neq 1$ is the value to which the base x is raised to yield u. The logarithms is read as "Logarithm of u to the base of x" or simply as "log base x of u". It is represented as follows.

$$\log_x u = v$$

Consequently, it can be written as

$$x^v = u$$

i.e., $$x^v = u \rightarrow v = \log_x u$$

Example: $2^3 = 8 \rightarrow \log_2 8 = 3$

$$3^{-4} = \frac{1}{8} \rightarrow \log_2 \frac{1}{81} = -4$$

It is apparent that logarithms with base 1 are not defined. Although there can be logarithms with number of bases, two of them are widely used.

Logarithms with base e = $2.718(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ are called natural logarithms or Napierian logarithms.

It can be written as $\log_e x$ or 1nx

5.4 Properties

5.4.1 Property I

$\forall x > 0$ and $x \neq 1$ $\log_x 1 = 0$

5.4.2 Property II

$\forall u > 0$ and $u \neq 1$, we have $\log_u u = 1$

5.4.3 Property III

$\forall n \in R, u > 0$, $x > 0$ and $x \neq 1$, we have $\log_x u^n = n \log_x u$

5.4.4 Property IV

Since logarithms are defined only for positive real numbers, if $u > 0$, $v > 0$, $x > 0$

and $x \neq 1$ $\log_x uv = \log_x u + \log_x v$

5.4.5 Property V

Since logarithms are defined only for positive real numbers, if $u > 0$, $v > 0$, $x > 0$ and $x \neq 1$

$$\log_x \frac{u}{v} = \log_x u - \log_x v$$

5.4.6 Property VI

For all $u > 0$, $v > 0$, $x > 0$ and $v \neq 1$, $x \neq 1$

$\log_x u \log_x u = \log_x u$ (change of base)

5.4.7 Property VII

For all $u > 0$, $v > 0$, $x > 0$ and $v \neq 1$, $x \neq 1$

$$\log_x u = \frac{1}{\log_u v}$$

5.4.8 Property VIII

For all $u > 0$, $x > 0$ and $x \neq 1$

$$x^{\log_x u} = u$$

5.4.9 Property IX

For all $a > 0$, $b > 0$, $x > 0$ and $x \neq 1$

$$a^{\log_x b} = b^{\log_x a}$$

5.4.10 Property X

$\forall m, n \in R, u > 0, x > 0, x \neq 1$ and $m \neq 1$, $n \neq 0$

$$\log_x n\, u^m = \frac{m}{n} \log_x u$$

5.4.11 Property XI

For all $u > 0, v > 0, x > 0$ and $x \neq 1$

For $x > 1$ If $u > v$

then $\log_x u > \log_x v$

For $0 < x < 1$

If $u > v$

then $\log_x u < \log_x v$

5.4.12 Property XII

For all $u > 0, v > 0, x > 0$ and $x \neq 1$

For $x > 1$ If $u < v$

then $\log_x u < \log_x v$

For $0 < x < 1$ If $u < v$

then $\log_x u > \log_x v$

5.4.13 Property XIII

$$\log_v u = \frac{\log_x u}{\log_x v} = \frac{\log u}{\log v}$$

5.5 Solved Problems

5.5.1 If n is a natural number such that

$$n = p_1^{\alpha_1} p_2^{\alpha_2} .....p_k^{\alpha_k} ; p_1, p_2......p_k \text{ are distinct primes}$$

Then prove that $\log n \geq k \log 2$

Sol: $n = p_1^{\alpha_1} p_2^{\alpha_2} ....p_k^{\alpha_k}$

$$\Rightarrow \log n = \alpha_1 \log p_1 + \alpha_2 \log p_2 + ... + \alpha_k \log p_k$$

$$\geq \alpha_1 \log 2 + \alpha_2 \log 2 + ... + \alpha_k \log 2 (\because p_i \geq 2)$$

$$= (\alpha_1 + \alpha_2 + ...\alpha_k) \log 2 \geq k \log 2 (\because \alpha_i \geq 1)$$

5.5.2 Solve $\log_{0.1} \sin 2x + \log_{10} \cos x = \log_{10} \frac{1}{\sqrt{3}}$

Sol: $\log_{0.1} \sin 2x + \log_{10} \cos x = \log_{10} \frac{1}{\sqrt{3}}$

$$\Rightarrow \log_{10^{-1}} \sin 2x + \log_{10} \cos x = \log_{10} \frac{1}{\sqrt{3}}$$

$$\Rightarrow -\log_{10} \sin 2x + \log_{10} \cos x = \log_{10} \frac{1}{\sqrt{3}}$$

$$\Rightarrow \frac{\cos x}{\sin 2x} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \sin x = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}$$

$$\Rightarrow x = n\pi + (-1)^n \frac{\pi}{3}, n \in Z$$

5.5.3 Find the value of $\log_{\sqrt{a}} \sqrt{a\sqrt{a\sqrt{a\sqrt{a\sqrt{a}}}}}$

Sol: $\sqrt{a\sqrt{a\sqrt{a\sqrt{a\sqrt{a}}}}} = a^{\frac{1}{2}} a^{\frac{1}{4}} a^{\frac{1}{8}} a^{\frac{1}{16}} a^{\frac{1}{32}} = a^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}} = a^{\frac{31}{32}}$

Therefore $\log_{a^{\frac{1}{2}}} a^{\frac{31}{32}} = \frac{31}{32}\frac{2}{1}\log a = \frac{31}{16}$

5.5.4 Find the value of $4^{2\log_9 3}$

Sol: $4^{2\log_9 3} = 4\log_9 3^2 = 4\log_9 9 = 4^1 = 4$

5.5.5 Find the value of $\sum_{r=1}^{89} \log_{10} \tan\frac{\pi r}{180}$

Sol: $\sum_{r=1}^{89} \log_{10} \tan\frac{\pi r}{180} = \log_{10}(\tan 1^0 \tan 2^0 \tan 3^0 ...... \tan 89^0)$

$= \log_{10}(\tan 1^0 \tan 89^0).....(\tan 44^0 \tan 46^0) \tan 45^0$

$= \log_{10}(\tan 1^0 \cot 1^0).....(\tan 44^0 \tan 44^0) \tan 45^0$

$= \log_{10}(1.1..... 1) = \log_{10} 1 = 0$

5.5.6 Find the value of $\sum_{1}^{n} \frac{1}{\log_{3^n} a}$

Sol: $\sum_{1}^{n} \frac{1}{\log_{3^n} a} = \sum_{1}^{n} \log_a 3^n$

$= \sum_{1}^{n} \log_a 3 = \log_a 3 \sum_{1}^{n} n$

$= \frac{n(n+1)\log_2 3}{2}$

5.5.7 If $\log_a(ab) = x$ then find the value of $\log_b(ab)$

Sol: Let $y = \log_b(ab)$

then $\frac{1}{x} + \frac{1}{y} = \log_{ab} a + \log_{ab} b = \log_{ab} ab = 1$

$$\frac{1}{y} = 1-\frac{1}{x} = \frac{x-1}{x} \rightarrow y = \frac{x}{x-1}$$

5.5.8 Solve $x+\log_{10}(1+2^x) = x\log_{10}5+\log_{10}6$

Sol: $x+\log_{10}(1+2^x) = x\log_{10}5+\log_{10}6$

$$x.\log_{10}10+\log_{10}(1+2^x)-x.\log_{10}5 = \log_{10}6$$

$$x.\log_{10}\frac{10}{5}+\log_{10}(1+2^x) = \log_{10}6$$

$$\log_{10}2^x(1+2^x) = \log_{10}6 \rightarrow 2^x(1+2^x) = 6$$

$$2^{2x}+2^x-6 = 0$$

$$\Rightarrow (2^x+3)(2^x-2) = 0$$

$2^x = -3$ *or* $2^x = 2 \Rightarrow 2^x = 2(\because 2^x > 0, 2^x = -3)$ is impossible) $\rightarrow x = 1$

Practice Problems

1. Given $\log_{10}2 = 0.3010$, find the value of $\log_{10}125$
2. Find the value of $\log_{22}256$
3. For $x \neq 2$, $\ln(x^2-2x)-\ln(x-2) = \ln(2x+5)$
4. Find the value of $\prod_{-45}^{45}\log(\cos x)$ where all measurements are in degrees.
5. Find n if $\log_2 5, \log_5 7, \log_7 9..........\log_n n+2 = 5$
6. If $\log_{10}343 = 2.5353$ then the find the least positive integer n such that $7^n > 10^5$

   (JEE problem)
7. If $4^{\log_9 2}+9^{\log_2 4} = 10^{\log_x 83}$ then find x.
8. If $\log_2\log_5\left(\sqrt{x+5}+\sqrt{x}\right) = 0$ then find x.
9. If $\log_{0.3}(x-1) < \log_{0.9}(x-1)$ then find the interval in which x lies.
10. If $2a^4 = b^4+c^4, abc = 8$ and $\log_b a, \log_c b, \log_a c$ are in G.P then find a, b, c.

5.6 Definition

Any common logarithm of a number can be expressed as a sum of a "characteristic" and "mantissa|. The integral part of the logarithm is called the "characteristic" adn the decimal part is called the "mantissa". Let us consider an example

$$\log_{10}2 = 0.30103$$

So, the characteristic of log 2 is 0 and the mantissa would be .30103

Let us calculate the characteristic and mantiss of $\log_{10}\left(\frac{1}{2}\right)$

$\log\left(\frac{1}{2}\right)$ = log 1 – log 2 = 0 0 – 0.30103 = – 0.30103 so that $\log\left(\frac{1}{2}\right)$ is negative.

Now it is found convenient that the mantissa of all logarithms should be kept positive. We therefore instead of – 0.30103 write –1 + 1 – 0.30103 = – + 0.69897. For shortness this latter expression is written as $\bar{1}.69897$. The horizontal line over 1 denotes that only the integral part is negative.

The logaritthms to the base 10 are called common logarithms or Briggs logarithms. $\log_{10}n$ is simply denoted by log n without specifying base.

$10^0 = 1 \Rightarrow \log_{10} 1 = 0$

$10^1 = 10 \Rightarrow \log_{10} 10 = 1$

$10^2 = 100 \Rightarrow \log_{10} 100 = 2$

$10^3 = 1000 \Rightarrow \log_{10} 1000 = 3$

We observe from the above that logarithm of a number between 2 and 10 lines between 0 and 1 i.e. 0+ some positive fraction between 0 and 1

Similarly, logarithm of a number between 10 and 100 lies between 1 and 2 i.e., 1+ some positive fraction between 0 and 1.

Simlarly, logarithm of a number between 100 and 1000 lies between 2 and 3 i.e., 2+ some positive fractionbetween 9 and 1.

So, we noticed that the logarithm of every positive real number conssts of an integral part and positive fractional part. The integral part of the logarithm of a number is called "characteristic" and the fractional part is called "mantissa".

Rule to find the characteristic of logarithm of a number

The characteristic of log7.25 is 0
The characteristic of log87.6 is 1
The characteristic of log7.25 is 2

Rule 1: If there are k digits in the integral part of a number n(n-1) then the characteristic of $\log_{10}n$ becomes k – 1

$10^0 = 1 \Rightarrow \log_{10} 1 = 0$
$10^{-1} = 0.1 \Rightarrow \log_{10} 0.1 = -1$
$10^{-2} = 0.01 \Rightarrow \log_{10} 0.01 = -2$
$10^{-3} = 0.001 \Rightarrow \log_{10} 0.001 = -3$

We observe from the above that logarithm of a number between0.1 and 1 lies between -1 and 0 i.e. -1+some positive fraction between 0 and 1

Similarly, logarithm of a number between 0.01 and 0.1 lies between -2 and -1 i.e.-2+some positive fraction between 0 and 1

Similarly, logarithm of a number between 0.001 and 0.01 lies between -3 and -2 i.e.-3+some positive fraction between 0 and 1

The characteristic of log0.584 is -1or $\bar{1}$

The characteristic of log0.04789 is -2 or $\bar{2}$

The characteristic of log0.002546 is -3 or $\bar{3}$

Rule 2: If there are k zeros after the decimal point in the in a positive number n(n<1) (i.e.a positive fraction) then the characteristic of $\log_{10} n$ is $\overline{k+1}$

Note: The characteristic of the logarithm of any number can always be determined by inspection

How to find the mantissa of logarithm of a number

The mantissa of a logarithm of a number is found from table of logarithms. Generally, we use four figure logarithm tables. There are tables of mantissa for any number of digits

Example: Find log 874.5

Sol: The characteristic of log 874.5 is 2(Since there are 3 digits in integral part of 874.5)

The significant digits in 874.5 are 8745. Se we pick 9415 opposite to 87 and under 4. Then we add 2 in the mean differences under 5 to 9415 which becomes 9417

Therefore log874.5 = 2.9417

Similarly, log0.0004=$\bar{4}.6021$=-4+0.6021

Tables of antilogarithms

We use tables of antilogarithms to know the value of x if log x is given.

We know the significant digits in x from the table of antilogarithms. Then we write the value of *x* by the characteristic of log x

Example

Find the value of x if log x =2.3654

From the table of antilogarithms, we pick 2317 opposite to .36 and under 5 and add 2 in the mean differences under 4 to 2317 which becomes 2319. The characteristic of log x is 2. So, there will be 3 digits in the integral part of x. Therefore $x = 231.9$

Example: Find the value of $\frac{25.62 \times \sqrt{346.5} \times \sqrt[3]{465.7}}{\sqrt{76.42} \times \sqrt[3]{44.44} \times \sqrt[4]{663.5}}$

$$k = \frac{25.62 \times \sqrt{346.5} \times \sqrt[3]{465.7}}{\sqrt{76.42} \times \sqrt[3]{44.44} \times \sqrt[4]{663.5}}$$

$$\log k = \log\left[\frac{25.62 \times \sqrt{346.5} \times \sqrt[3]{465.7}}{\sqrt{76.42} \times \sqrt[3]{44.44} \times \sqrt[4]{663.5}}\right]$$

$$= \log 25.62 + \frac{1}{2}\log 346.5 + \frac{1}{3}\log 465.7 - \frac{1}{2}\log 76.42 - \frac{1}{3}\log 44.44 - \frac{1}{4}\log 663.5$$

$$= 1.4085 + \frac{1}{2}(2.5397) + \frac{1}{3}(2.6682) - \frac{1}{2}(1.8832) - \frac{1}{3}(1.6478) - \frac{1}{4}(2.8218)$$

$$= 1.4085 + 1.2699 + 0.8561 - 0.9416 - 0.5493 - 0.7055 - 2.1964 = 1.3381$$

$$k = \text{antilog } 1.3381 = 21.78$$

Example If $\log_{10} 2 = 0.3010$ then find the number of digits in 256^{10}

Sol: $k = 256^{10} = (2^8)^{50} = 2^{400}$

then $\log k = 400 \log 2 = 400 \times 0.3010 = 120.4$

Number of digits in k = 121